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# Chaos and Unpredictability with Time Inconsistent Policy Makers*

**Abstract**

We study the existence of equilibria with complex dynamics in a policy problem with time inconsistent policy makers. We consider an economy where a policy maker every period selects the level of a durable public good (or bad) that strategically links policy making periods. When the decision process is time consistent, as when a benevolent planner selects the policy, the economy has a unique equilibrium in which the state converges to a deterministic steady state. When the decision process is not time consistent, as in a political equilibrium, equilibria with complex cycles and aperiodic, chaotic dynamics exist under easily satisfied conditions. Depending on the fundamentals of the economy, these equilibria may generate ergodic distributions that consistently overshoot the planner's steady state, or that fluctuate around it. The size of the support of the cycles and the chaotic region depends on the degree of time inconsistency: as the degree of time inconsistency converges to zero, the support of the cycles or the chaotic behavior converges to zero as well.

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# 1 Introduction

Environments characterized by dynamic inconsistencies are pervasive in economics. They not only include settings with time inconsistent preferences, but also many natural strategic interactions in which the players have "standard" exponential preferences, such as voluntary contribution games and common pool problems (see Levhari and Mirman [1980], Feistman and Nitzan [1991], Marx and Matthews [2000], Matthews [2013], Battaglini et al. [2014], among others); or various political economy problems such as positive models of public debt (see Persson and Svenson [1989], Alesina and Tabellini [1990], Lizzeri [1999], Battaglini and Coate [2008], Yared [2010] among others) or of durable public goods (Battaglini and Coate [2007]). In dynamic political economy games, time inconsistency in decision making emerges from the fact that decision-makers change over time and have heterogeneous preferences; in voluntary contribution and common pool games, time inconsistency stems from externalities in the individual decisions of multiple decision makers.[1]

Despite the importance of these problems, there is still a limited understanding of the strategic behavior in stochastic games with time inconsistency. In applied work, the focus has been either on simplified environments such as, for example, models with finite horizons or models in which the players have dominant strategies, where equilibria are more readily characterized; or on environments in which well-behaved equilibria exist and can be relied upon to make predictions. The equilibria in these games often look like solutions to a constrained planner's problems, in which the state variable monotonically converges to a steady state –albeit too low as in dynamic public good games, or too high, as in political economy models of public debt.[2]

In this paper we study a simple dynamic policy game with time inconsistency and we ask

[1] In all the example presented above a decisionmaker faces a different intertemporal rate of substitution between $t+1$ and $t+2$ at $t+1$ than a decision maker at $t$. A more detailed discussion of the relationship between these models and single agent models with time inconsistent preferences will be presented in Section 5.2.

[2] Characterizations of political economy games or public policy games with time inconsistency as constrained planner's problems have been provided under alternative assumptions by, among others, Battaglini and Coate [2008], Klein et al. [2008], Yared [2010].

whether it can generate equilibria with complex dynamics. Specifically, we aim at characterizing conditions under which time inconsistency allows for the existence of equilibria with cycles (with possibly arbitrarily long periods) and equilibria in which the state variables follow chaotic trajectories. In these equilibria the state variable can be said to be unpredictable in the sense that it is sensitively dependent on the initial state: an arbitrarily small perturbation on the initial state can generate a large deviation in the long term even in the absence of shocks. If the initial state is observed with noise, as is natural to assume, then long term behavior is unpredictable even if the noise is arbitrarily small.

To analyze the issue, we study an infinite horizon game in which an incumbent policy maker selects a one-dimensional policy in every time period: a higher level of the policy improves the state variable, a public good that increases the utility of all citizens in future periods (say, the state of the environment); the policy, however, reduces stock resources that can be used for public expenditures that would favor the incumbent's constituency. We study the political equilibrium of this economy assuming that two parties alternate in power as in the classic game by Alesina and Tabellini [1990]. As we explain in detail below, the logic of the model can be applied with minor changes to other economic environments, such as single agent consumption savings decision problems with quasi-hyperbolic discounting; dynamic free rider problems, such as voluntary contribution and common pool games; or more sophisticated political economy games with non cooperative bargaining.

In the absence of time inconsistency (such as when the policy-maker is a benevolent planner), the economy has a unique equilibrium with simple dynamics, where the state variable monotonically converges to a unique steady state. When policies are selected in a political equilibrium, however, the set of equilibria is very different and equilibria with complex dynamics may emerge. We first focus the analysis to the case in which preferences are quasilinear. We show that in the presence of any degree of time inconsistency we always have Markov equilibria with persistent cycles and aperiodic, chaotic dynamics if the relative importance of the public good component is sufficiently large or, ceteris paribus, if time inconsistency is not too large. We then extend the analysis to a more general class of utility functions.

In the equilibria with chaos, starting from almost any initial condition, the state variable converges to a region with positive measure and then "wanders around" in its inside, following an aperiodic trajectory. Under general conditions, the state variable behaves like a random variable with a continuous distribution, in the sense that the ergodic distribution of the state variable converges to an absolutely continuous distribution on the set (a phenomenon often referred to as "ergodic chaos"). For some equilibria, this absolutely continuous distribution can be characterized in closed form as a function of the parameters of the economy (such as the degree of time inconsistency and the importance of the public good).

Existence and the properties of these equilibria are intimately connected to the presence of time inconsistency. The size of the set in which the state wanders around depends on the degree of time inconsistency: as time inconsistency converges to zero, chaotic equilibria continue to exist, but the size of the set in which the state wanders around shrinks to zero. A similar phenomenon occurs for equilibria with cycles: as time inconsistency converges to zero, the distance between the states in the periods converges to zero. This implies that depending on the degree of time inconsistency, equilibria with complex dynamics can be consistent with a variety of phenomena: from describing environments with large, possibly cyclical policy swings, when time inconsistency is large; to environments with small, noisy perturbations around a steady state generating a "fog of predictions" around an expected long term value, when time inconsistency is small. Failure to converge to a deterministic outcome in these equilibria highlights a novel source of inefficiency, distinct from the typical inefficiencies highlighted in the literature, where steady states are usually characterized as having too little of a good thing or too much of a bad thing. We show that when there are equilibria with persistent cycles, or equilibria with chaos, we do not necessarily have a simple "one dimensional" bias. It is possible to construct equilibria under which the state of the economy fluctuates around the planner's optimum. This outcome is even worse than reaching a constant steady state equal to the inefficient expected value since preferences are concave.

A limitation of the results described above is that the chaotic behavior we characterize is not typical of all equilibria of our dynamic economy, but it is instead a feature of the specific class of equilibria whose existence we prove. Our results can be collectively interpreted as an *impossibility result*: for the simple yet natural economy we consider, it is impossible to predict equilibrium behavior in the sense that there are always chaotic equilibria that make deterministic predictions impossible even in the absence of shocks.

The paper contributes to three main lines of research. The question of whether dynamic economic models can generate complex dynamics and chaos has been studied extensively in the 80s and 90s. Identifying natural examples of economic environments with robust chaotic dynamics occurring in equilibrium, however, has proven to be an elusive task. Early economic examples of models with complex dynamics relied on a sufficient condition proposed by Li and Yorke [1975] that, while relatively easy to establish, did not exclude the typical case in which trajectories with complex dynamics are unstable and reachable only from a measure zero of initial conditions, making complex cycles or chaotic behavior unobservable (this form of unobservable chaotic behavior is sometimes refereed to as topological chaos).[3] Economic examples with more robust forms of complex dynamics, such as the ergodic chaos mentioned above, have been presented in the subsequent literature. These applications, however, relied on sufficient conditions that require the difference equation describing the dynamics to assume specific functional forms: piecewise linear maps, typically "V" or "inverse V" shaped; or piecewise smooth, expansive maps (i.e. functions with nondifferentiable "spikes" and derivative larger than one in absolute value on both side of

[3] Early seminal work include Day [1982], Benhabib and Day [1983], Grandmont [1985], who study overlapping generation models; Montrucchio [1982], Boldrin and Montrucchio [1983], Matsuyama [1999], Khan and Mitra [2005, 2020], who study growth models; and Bewley [1986] and Woodford [1988] who study models with market impefections. A critique of the notion of topological chaos used in some of the early literature is presented by Grandmont [1985] and Melese and Transue [1986]. See Baumol and Benhabib [1989] and Majumdar et al. [2000] among others for surveys of this literature.

the "spike"). These properties are not naturally derived for optimal investment functions, except if specific technologies, or exogenous constraints are imposed, such as credit constraints or other constraints that force the state variable to become non monotonic.[4] While there are environments in which these assumptions are appropriate, our results rely neither on the assumptions nor on the techniques used in these works. Our contribution relies on the construction of a novel class of equilibria, one that is large enough to select an equilibrium with the right properties for complex dynamics, in each parametrization of the environment. The characterization, in turn, is a consequence of the presence of time inconsistency. To our knowledge, this is the first paper to establish a link between the presence of time inconsistency and the presence of complex cycles and chaos.

The second literature to which our paper contributes is the literature on dynamic decision with time inconsistency. Phelps and Pollack [1968] started this literature by characterizing the linear Markov equilibrium in a single agent model of intertemporal consumption allocation. A general characterization of the Euler equations in similar problems is presented in Harris and Laibson [2001], and analyses of saving dynamics for a representative agent with quasi-geometric preferences are presented by Morris and Postlewaite [1997], Chatterjee and Eyigungor [2014], Cao and Werning [2018], among others. Krusell and Smith [2003] also studied the issue of multiplicity of equilibria in these problems, and highlighted indeterminacy due to multiple possible equilibrium steady states. The problem we study in this paper is different: while we also obtain multiple equilibria, our main result is the proof of the existence of equilibria with complex dynamics. In these equilibria, unpredictability occurs for a given equilibrium, not because of multiplicity.

Finally, our work contributes to the study of applied models where time inconsistency emerges

[4] Economic examples with these properties have been presented, for example, by Day and Shafer [1987], Day and Pianigiani [1991], Deneckere and Judd [1990] and more recently Matsuyama et al. [2016]. In these examples, the state $x_t$ evolves according to a system $x_{t+1} = y(x_t)$ in which $y(\cdot)$ is assumed either piecewise linear and "V" shaped; or to be "expansive," i.e. satisfying $\inf(y'(x)) > 1$, so that if there is a maximum, it occurs in a non differentiable "spike." These results rely on sufficient conditions first presented by Lasota and Yorke [1973] and/or its subsequent refinements.

in equilibrium despite players having standard exponential preferences. We have already cited above works on voluntary contribution and common pool games and political economy. As mentioned, this literature has mostly focused on well behaved equilibria or environments in which the state converges to a deterministic steady state. Among the exceptions, Boyland, Ledyard and McKelvey [1996] and Bai and Lagunoff [2011] study the problem in a political economy setting, while Battaglini, Palfrey and Nunnari [2012] study voluntary contributions to a public good.[5] Boyland et al [1996] considers a model in which simple cycles with finite orbit may emerge when the policy-maker selects policies that can be defeated by the smallest possible majority, and s/he can commit for at least 3 periods into the future; the length of the commitment period determines the length of the cycle in this model.[6] Bai and Lagunoff [2011] study a dynamic political game in which policies at $t$ affect political turnover at $t+1$. They show conditions under which the equilibrium may converge to a stable steady state following a dampened cycle. Battaglini et al. [2012] considers a model of free riding in which $n$ agents independently contribute to a public good: using numerical examples, they show the existence of Markov equilibria with dampened cycles and with cycles of period 2.[7] None of these papers, nor to our knowledge any other in the political economy literature, has linked political distortions and, more generally, time inconsistency in dynamic decision making to the existence of complex limit cycles and/or chaotic behavior.[8]

[5] Among other recent works studying endogenous cycles in political economy models there are Krusell and Rios Rull [1996], Acemoglu and Wolitzky [2014] and Dovis et al. [2016].

[6] The authors also show that no cycles is possible if the policy makers can not commit to a policy.

[7] The NBER working paper Battaglini et al. [2012] was published as Battaglini et al. [2014], but some of the results on cyclical equilibria were omitted in the 2014 version.

[8] A different (and less related to our work) body of research is the literature on the so called "political business cycles." This literature looks at models in which fluctuations in economic activity are generated by recurrent stimuli right before an election by an incumbent attempting to signal his/her competence to influence the electoral outcome; or right after an election as the uncertainty on the type of the winning party is resolved. These are typically stationary models with no underlying state variable, in which fluctuations coincide with the electoral cycle, not with a long term evolution of a state variable. See Alesina [1988], among others, for a survey.

## 2 Model

Consider an economy in which two parties alternate in power, call them $A$ and $B$. Each party is associated to a constituency of citizens. We assume that there is a continuum of citizens and we normalize the size of the constituency of each party to one. The party in power at time $t$ selects a policy $p_t$ from a set of feasible policies. The policy generates immediate costs or benefits for the citizens, and it also contributes to a long term state variable $x_t$ that also affects the citizens' utility. For example, $p_t$ may be a polluting activity (say fracking or a new coal power plant) that generates economic benefits to all (or a subset of) the citizens, yet contributes to global warming (as measured by $x_t$).

The state variable $x_t$ takes values in a compact set $X$ and evolves according to:

$$x_{t+1} = (1 - \gamma)x_t + p_t, \tag{1}$$

where $\gamma$ is the rate of depreciation of $x_t$. The policy $p_t$ takes values in the set $P = [-l, \infty)$ for some $l \geq 0$, and must be such that the state $x_{t+1}$ satisfies $x_{t+1} \geq (1 - \gamma)x_t - l$ and remains in $X$. The lowerbound reflects the fact that there may be limits on the feasibility of a reduction in the state $x$. The set of feasible states $X$ is large, so it should not be expected to be binding in equilibrium (yet it may be binding out-of-equilibrium).

In every period $t$, party $l \in \{A, B\}$ has a probability 1/2 to be in power. This assumption reflects the idea that the two parties have the same constituency, so the identity of the majority party at $t$ is determined by chance. Citizens are assumed to be identical, except for the party whose constituency they belong to. Define

$$u^{i,j}(p, x) = u^{i,j}(y - (1 - \gamma)x, x) \tag{2}$$

as the indirect utility function of a citizen in the constituency of party $i$ when the party in office is $j$, the state at the beginning of the period is $x$, the state at the end of the period is $y$, and thus the policy is $p = y - (1-\gamma)x$. A policy $p$ can be interpreted as the expenditure on local public goods, subsidies or other policies that the party in office can target to its constituency. Utility (2) depends on the party in power $j$ because, even if the levels of expenditure is the same for the two parties, the policy mix chosen by each party would naturally be different. The function $u^{i,j}(p,x)$ is assumed to be concave and continuously differentiable in $p$ and $x$, with derivative with respect to the $l$th component equal to $u_l^{i,j}(\cdot)$ for $l = 1,2$. We assume $u_1^{i,i}(\cdot) > u_1^{i,j}(\cdot)$ and $u_1^{i,i}(\cdot) > 0$: citizens' marginal utility for policies targeted to them is positive, and they derive higher utility from policies targeted to them than from policies targeted to the other constituency. The spillover of policies by policy maker $j$ (targeted to district $j$) on district $i$ can be positive (i.e., $u_1^{i,j}(\cdot) > 0$), as in the case of an highway or bridge; or negative (i.e., $u_1^{i,j}(\cdot) < 0$), as in the case of a polluting power plant that benefits only to $j$'s constituency, but that yet pollutes the air of both $i$'s and $j$'s constituencies. A specific example of these indirect utility functions is described below.

In this economy, an allocation is described by an infinite sequence $x_\infty$ where $x_\infty = (x_0, ..., x_t, ...)$ and $x_0$ is exogenously given. The intertemporal utility at $t = 1$ of an agent in party $i$'s constituency is:

$$U^i(x_\infty) = \sum_{t=1}^{\infty} \delta^{t-1} \left[ u^{i,\iota(t)} \left( x_t - (1-\gamma)\, x_{t-1}, x_{t-1} \right) \right],$$

where $\delta$ is the discount factor, $\iota(t)$ is the incumbent party in office at time $t$.

In Section 3 we study the case in which preferences are quasi-linear and separable in $x$ and $p$. Specifically, we assume the per period utility function:

$$u^{i,j}(p,x) = \alpha_{i,j} K \cdot p - e(x), \tag{3}$$

where $K$ is a strictly positive constant. We assume that $\alpha_{i,j} = 1$ when $i = j$ and so $i$'s party is in office; and $\alpha_{i,j} = \alpha < 1$ when $i \neq j$ and $i$'s party is not in office. The parameter $\alpha$ is a direct measure of the time inconsistency generated by the political alternation of power. When $\alpha = 1$ there is no political conflict and no time inconsistency, since the policies of the two parties have the same effects on all citizens.[9] When $\alpha \in [0, 1)$ the policy benefits both the constituency of the party in power and the constituency of the party out of power, albeit less for the latter if $\alpha < 1$. When $\alpha \in (-\infty, 0)$, instead, the policy benefits the constituency of the party in power, but generates negative externalities for the rest of the citizens.[10] In Section 3, we also assume a quadratic cost function $e(x) = (\beta/2)(x - \widehat{x})^2$, but we later relax this assumption in Section 4.

For most of the analysis we focus on symmetric Markov perfect equilibria, in which the parties use the same strategy, and in each period $t$ these strategies are time-independent functions of the state $x_t$. Non-Markovian strategies will be discussed in Section 4.[11] A Markovian strategy is a function $p(x)$, where $p(x)$ is the policy of the party in power when the state is $x$. Naturally, once $p(x)$ is defined, then the state variable at $t+1$ is automatically defined as: $y(x) = (1-\gamma)x + p(x)$. In the following it will be more convenient to define equilibria in terms of $y(x)$. For the remainder of the paper we refer to $y(x)$ as the *investment function.* Associated with any Markov perfect equilibrium of the game is a value function, $v(x)$, which specifies the expected discounted future payoff to an agent when the state is $x$.

An equilibrium $y(x)$ and a support $X$ for the state variable $x$ define a dynamical system. We are interested in studying the dynamics that can emerge in equilibrium. It is worth stressing that the dynamics of the state in a symmetric equilibrium is deterministic and fully determined by the equilibrium $y(x)$. The two parties in power alternate in power with probability 1/2, but they adopt the same strategy $y(x)$ in equilibrium, so the evolution of $x$ does not depend on the outcome of the election but only on the initial condition $x_0$ and the shape of the transition function

[9] The fact that with $\alpha = 1$ we do not have time inconsistency does not imply that policies are Pareto efficient: even with $\alpha = 1$ the policy maker in office ignores the externality on the constituency of the other policy maker.

[10] The indirect utility function (3) has a simple microfoundation. Assume that there are two possible policies: $p^L$ that generates a marginal utility $K$ on party $L$'s constituency and $\alpha K$ on $R$'s constituency; and a symmetric $p^R$ that generates a marginal utility $K$ on party $R$'s constituency and $\alpha K$ on $L$'s constituency. In a Markov equilibrium, for any level of expenditure $p$, party $i$ would spend all in $p^i$, implying (3).

[11] The main result of our analysis is in proving the existence of equilibria with cycles, and/or unpredictable and chaotic behavior. The focus on Markov equilibria, therefore, is without loss of generality and makes the results stronger as it relies on simpler strategies.

$y(x)$[12] . We define $y^0(x) = x$, $y^1(x) = y(x)$ and the $k$th iterate $[y]^k(x)$ as $[y]^k(x) = y([y]^{k-1}(x))$. For any starting condition $x_0$, iteration of $y(x)$ naturally define an orbit $\{x_0, x_1, ...x_k, x_{k+1}, ...\}$ in which $x_k = [y]^k(x_0)$. A cycle is a set $\{x_0, x_1, ..., x_\tau\}$ such that $x_k = y(x_{k-1})$ for all $k \geq 1$ and $x_0 = y(x_\tau)$. Any element of a cycle with $\tau$ element satisfies the condition $x_k = [y]^{\tau-1}(x_k)$ and is called a periodic point of period $\tau$. The simplest, and most widely studied, case of cycle is a *steady state*, which is a cycle of period 1. We define a cycle $\{x_0, x_1, ..., x_\tau\}$ to be stable if for all points $x_k$ in the cycle there exists an open neighborhood $U$ of $x_k$ such that for all $x \in U$, we have $y^{m\tau}(x) \in U$ for any integer $m \geq 1$ and $\lim_{m\to\infty} y^{m\tau}(x) = x_k$. When a cycle is stable a small perturbation to the state variable does not alter the long run behavior of the system.

An orbit is aperiodic if there is no finite cycle. An orbit that is aperiodic is not necessarily "chaotic" in an intuitive sense. For example, an orbit that converges to a deterministic steady state is aperiodic, but has a well definite and predicable limit behavior. The same is true for an orbit that converges to a stable cycle of period $k > 1$. The mathematical literature has offered various definition of "chaotic" behavior of a deterministic dynamical systems. The intuitive features "chaotic" orbits in as set $I$ are that: a. the system is irreducible and aperiodic in $I$; b. it has a complex orbit reaching almost all states in $I$; and it has sensitive dependence, meaning that even an arbitrarily change in the initial condition leads to a large deviation in the long term. Dynamical systems with these properties are said to be incomputable or unpredictable in the sense that they give different predictions for arbitrarily close initial conditions (See, for example, Devaney [1989]).

A standard formal definition of chaotic behavior is provided by Devaney [1989]. We say that an investment function $y$ is *transitive* in $I$ if for any open $U$, $V \subset I$, there exists a $k$ such that $y^k(U) \cap V \neq \emptyset$. Intuitively, a topologically transitive map "wanders" around the support, moving

[12] The fact that parties alternate in power stochastically is important only to the extent that it generates dynamic time inconsistency. In Section 5 we present an alternative decision model with a single decision maker with hyperbolic discounting and no shocks, and a free rider model with $n$ players and again no shocks as examples in which time inconsistency emerges with no shocks.

under iteration from one arbitrarily small neighborhood to any other. An investment function $y$ exhibits *topological chaos* in a set $I$ if it is transitive and it has a set of periodic points that is dense in $I$. If the two conditions of this definition are satisfied, then it can be shown that $y$ is sensitive on initial conditions in the sense that there exists a $\delta$ such that, for any $x \in I$ and any neighborhood $N$ of $x$ there exists a $z \in N$ and a $m \geq 0$ such that $|y^m(x) - y^m(z)| > \delta$.[13]

An alternative approach to define chaotic behavior consists in directly looking at the long term behavior of the state variable. A dynamical system is said to display *ergodic chaos* if the system is ergodic and the unique invariant distribution of the Perron-Frobenius operator is absolutely continuous with respect to the Lebesgue measure. This definition implies that, starting from a generic initial condition, the orbit described by $y$ "fills up" the entire support of the ergodic distribution and thus defines extremely complex dynamics. As we will see, our equilibria will satisfy both the topological and the ergodic definition of chaos. We will formally define and discuss ergodic chaos in Section 3.2.

Before studying the equilibrium described above, it is useful to characterize the optimal policy when it is selected by a utilitarian planner who maximizes the sum of utilities for all citizens (henceforth, the planner) as a benchmark. Define the feasible set as $F(x;\gamma,l) = \{y \in X \,|y \geq (1-\gamma)x - l)\,\}$. The planner solves the following problem:

$$V(x) = \max_{y \in F(x;\gamma,l)} \{\Gamma(x,y;\alpha,\gamma) + \delta V(y)\} \tag{4}$$

where $\Gamma(x,y;\alpha,\gamma) = (1+\alpha)\,K \cdot (y - (1-\gamma)x) - 2e(x)$ and $V(y)$ is the planner's continuation value function. Note that $\Gamma(x,y;\alpha,\gamma)$ is continuous, differentiable in $y \in X$ for a given $x \in X$, and concave for $x, y \in X$, strictly with respect to $x$ alone. By a standard argument, we can show that there exists a unique $V^*(x)$ satisfying (4) that is strictly concave and differentiable. In the quasilinear environment with quadratic $e(x)$ described above, the optimal policy $Y^*(x)$ that solves

[13] Devaney [1989] originally included sensitive dependence on initial conditions in the definition of topological chaos; Banks et al. [1992] subsequently proved that it is implied by transitivity and a dense orbit.

(4) is also uniquely defined and admits a unique steady state: $Y^* = \widehat{x} + \frac{1+\alpha}{2\beta}\left(\frac{1}{\delta} - (1-\gamma)\right)K$. The state variable monotonically converges to $Y^*$ for any initial condition $x_0$.

# 3 Political Equilibrium

## 3.1 Existence of periodic and aperiodic equilibria

We now turn to the study of the equilibria of the game in which policies are chosen by the incumbent party (henceforth, the incumbent) under the assumption of quasi-linear preferences (3). The goal of this subsection is to prove the existence of equilibria with cycles or aperiodic dynamics. The exact type of dynamics that is possible is studied in the next subsection.

The incumbent's problem can be written as follows:

$$\max_{y \geq (1-\gamma)x - l} \left\{K\left[y - (1-\gamma)x\right] - e(x) + \delta v(y)\right\}. \tag{5}$$

The incumbent maximizes the expected utility of her constituency taking $v\left(y\right)$ as given, thus ignoring the cost/benefit for the constituency of the other party, $\alpha K\left[y - (1-\gamma)x\right] - e(x)$. In equilibrium, the expected continuation in state $x \in X$, $v(x)$, must satisfy:

$$\begin{aligned} v(x) \quad &= \quad \frac{1}{2}\left[K(y(x) - (1-\gamma)x) + \delta v(y(x))\right] \\ &\quad + \frac{1}{2}\left[\alpha K(y(x) - (1-\gamma)x) + \delta v(y(x))\right] - e(x). \end{aligned} \tag{6}$$

If $x$ is the state, each party suffers a disutility $e(x)$ for sure; with probability 1/2 the incumbent remains in office and selects $y(x)$, obtaining $K(y(x) - (1-\gamma)x) + \delta v(y(x))$; with probability 1/2 the party is no longer in office and receives only $\alpha K\left(y(x) - (1-\gamma)x\right) + \delta v(y(x))$, since the policy $y(x)$ is selected by the other party. An equilibrium is characterized by a pair of functions $y(\cdot)$ and $v(\cdot)$ such that for all states $x$, $y(x)$ solves (5) given $v(x)$ and $v(x)$ solves (6) given $y(x)$.

The incumbent's trade-off can be described as follows. By increasing $y$, s/he increases current utility for his/her district; by increasing $y$, however, s/he also reduces future's utility for all through the effects on the expected continuation function $v(y)$. There are two key differences between (5)-(6) and the planner's problem (4). The first is that, as we mentioned above, in any given period the incumbent selects a policy that maximizes the expected utility of his/her constituency alone, ignoring the spillover effects on the constituency of the other party. The second (and most important) difference is that the value of the incumbent's problem (5) does not coincide with the incumbent's continuation value function (6) except in the special case in which $\alpha = 1$. The value of (5) is the expected value for the incumbent; in the continuation of the game, however, the incumbent at $t$ will remain incumbent only with probability $1/2$. This feature make the incumbent's problem time inconsistent since her objective function when selecting the policy does not coincide with the expected continuation value.

In the planner's solution the marginal effect of the state on the expected continuation value is independent of expected future policy $y(x)$:

$$
\begin{aligned}
\left[V^*\right]'(x) &= -(1+\alpha)(1-\gamma)K - 2e'(x) + \left[(1+\alpha)K + \delta\left[V^*\right]'(Y^*(x))\right]\left[Y^*\right]'(x) \\
&= -(1+\alpha)(1-\gamma)K - 2e'(x)
\end{aligned}
$$

where $\left[Y^*\right]'(x)$ is the derivative of the planner's policy function and the second equality follows from the envelope theorem. In the political equilibrium, however, the standard envelope theorem is not directly applicable, making the optimal decision for the incumbent critically dependent on her expectation of future behavior of the other party. The incumbent's value function (6) can be written as:

$$
\begin{aligned}
v(x) &= \left[K\left(y(x) - (1-\gamma)x\right) + \delta v(y(x))\right] - e(x) \\
&\quad -\frac{1}{2}(1-\alpha)K \cdot \left[y(x) - (1-\gamma)x\right]
\end{aligned} \tag{7}
$$

where the first line on the right hand side is the objective function that is maximized by the incumbent at $t+1$, and the second line collects the wedge between the incumbent's objective function and the expected continuation value. Applying the envelope theorem to the first term in

(7),[14] we have:

$$v'(x) = -e'(x) - (1+\alpha)K(1-\gamma)/2 - (1-\alpha)Ky'(x)/2. \tag{8}$$

The key feature of this expression is that the marginal change in the value function depends on the expected policies selected by future incumbents, i.e. $y(x)$. If the incumbent at $t$ expects the incumbent at $t+1$ (herself or the opponent) to rapidly increase $y$ as a function of $x$ (i.e. a high positive $y'(x)$), then she will have higher incentives to keep the state $x$ low at $t$; similarly, if s/he expects the future incumbent to clean up $x$ or increases it slowly (i.e. a low or negative $y'(x)$), then she will have higher incentives to pollute. The important question for predicting behavior in a political equilibrium is what kind of expectations on $y(x)$ are consistent with equilibrium behavior.

In equilibrium the policy must solve (5). From the first order necessary condition we have $K = -\delta v'(z)$ for $z = y(x)$. Ignoring the policy constraint for the moment, an interior optimum satisfies both this condition and (8). Combining the conditions we obtain:

$$\begin{aligned} K/\delta &= e'(z) + (1+\alpha)K(1-\gamma)/2 + (1-\alpha)Ky'(z)/2 \\ &\Leftrightarrow \; y'(z) = \left[2/\delta - (1+\alpha)(1-\gamma) - 2e'(z)/K\right]/(1-\alpha). \end{aligned} \tag{9}$$

This is a simple differential equation that, under the assumption that $e(z) = (\beta/2)(z-\widehat{x})^2$, can be solved in closed form up to a free constant $c$. The solution can be written as:

$$y(z,c) = \frac{2/\delta - (1+\alpha)(1-\gamma) + 2\beta\widehat{x}/K}{1-\alpha} \cdot z - \frac{\beta z^2}{(1-\alpha)K} + c. \tag{10}$$

The following Proposition 1 characterizes a sufficient condition such that an equilibrium exists in which the investment function coincides with (10) in all periods, except for at most a finite

[14] The assumption of differentiability here is without loss of generality since, as we will show in Proposition 1, the equilibrium is almost everywhere differentiable (and always in the relevant region). We assume here differentiability only for ease of notation, since the same argument regarding the dependence of differentials of $v$ on $y$ can be made without assuming differentiability.

transition period where the policy must accommodate the feasibility constraint.

To see the logic behind the equilibrium construction of Proposition 1, note that the policy function $y(\cdot, c)$ defined in (9) is such that the marginal increase in utility at $t$ of a policy (i.e., the left hand side of the first line in (9)) is exactly equal to the marginal change in expected utility in periods following $t$ (i.e., the right hand side of the first line in (9)). This makes the policy maker's objective function flat with respect to the new state $y$ at $t + 1$. When a party expects future policy makers to invest according to a function $y(\cdot, c)$ as in (10), therefore, the incumbent is indifferent regarding which policy to choose: implying that following the strategy $y(\cdot, c)$ in state $x$ is indeed optimal. If both parties expect $y(\cdot, c)$ to be used in all future states, then $y(\cdot, c)$ defines an equilibrium analogous to a mixed strategy equilibrium, where indifference is generated by the rate of future investments, but relying on pure strategies.

The construction, however, requires some care, since $y(\cdot, c)$ violates the feasibility constraint $y \geq (1 - \gamma)x - l$ in states that are sufficiently large or sufficiently small.[15] Proposition 1 shows that, in these cases, $y(\cdot, c)$ can be adjusted to accommodate for feasibility. After the adjustment the dynamics may differ from the dynamics described by $y(\cdot, c)$, but only for a finite transition period (in which the state is at a corner solution with $y = (1 - \gamma)x - l$). The key observation is that there is a set $X^*$ in which $y(\cdot, c)$ is a self map: once $x \in X^*$, then $y(x, c) \in X^*$, so once the orbit enters $X^*$, it never exits. The adjustment, moreover, is such that once $x \in X^*$, no policy maker has incentives to deviate outside $x \in X^*$.

To formally state Proposition 1, define $R$ to be the ratio $R{=}\beta/K$. This ratio captures the temptation for an incumbent to abuse its position in selecting the policy. The numerator measures the importance of the externality generated by $x$ on society; the denominator measures the

[15] Indeed, for any $c$, we have $\lim_{x \to \pm\infty} [y(x, c) - (1 - \gamma)x + l] < 0$.

importance of the private benefit of the policy for the incumbent. Define the threshold:

$$R^{*}(\alpha)=\frac{4\delta(1-\alpha)(2-\gamma)+\delta(1+\alpha)(1-\gamma)\gamma-2\gamma}{2\delta(\widehat{x}\gamma+l)} \tag{11}$$

and the sets:

$$\begin{aligned}\mathcal{C}^{*} &= \left[\frac{(3-\varphi_1)(1+\varphi_1)}{4\varphi_2},\frac{(4-\varphi_1)(2+\varphi_1)}{4\varphi_2}\right]\\ \text{and } X^{*}(c) &= \left[\frac{\varphi_1-1-\sqrt{(\varphi_1-1)^2+4\varphi_2 c}}{2\varphi_2},\frac{\varphi_1^2+4\varphi_2 c}{4\varphi_2}\right]\end{aligned} \tag{12}$$

where we denote $\varphi_1=\frac{1}{1-\alpha}\left[\frac{2}{\delta}-(1+\alpha)(1-\gamma)+\frac{2\beta}{K}\widehat{x}\right]$ and $\varphi_2=\beta/[(1-\alpha)K]$, the coefficients of, respectively, the linear and quadratic term in (10). We say that an orbit *generically converges* to a cycle if it converges to a cycle for all initial states $x^0\in X$, except at most for a subset of measure zero. We have:

**Proposition 1.** *Consider an economy with $R\geq R^{*}(\alpha)$:*

- *For any $c^{*}\in\mathcal{C}^{*}$, there is an equilibrium $y^{*}(x,c^{*})$ in correspondence of which the state variable is in $X^{*}(c^{*})$ for all periods except at most for a finite transition period, and such that $y^{*}(x,c^{*})=y(x,c^{*})$ as defined in (10) for all $x\in X^{*}(c^{*})$.*
- *For any $c^{*}\in\mathcal{C}^{*}$, moreover, each equilibrium $y^{*}(x,c^{*})$ defines an orbit that either generically converges to a stable cycle of finite period $m\geq 2$ in $X^{*}(c^{*})$, or that is aperiodic in $X^{*}(c^{*})$.*

The first part of Proposition 1 shows that (10) essentially describes an equilibrium of our economy (where the behavior may not be described by (10) only, at most, for a finite number of periods); and the second part shows that all these equilibria do not converge to a deterministic steady state. This result should be contrasted with the planner's solution described in the previous section, in which we have a unique equilibrium converging to a deterministic steady state. Concrete examples of equilibria with complex cycles or aperiodic dynamics will be presented in the next section, after we fully characterize the dynamics of the equilibria of Proposition 1.

A key feature of the equilibria constructed in Proposition 1 is that the investment function $y(x)$ is non-monotonic and hump-shaped (see for example Figure 1). This shape naturally emerges from the equilibrium condition (9) discussed above when $e(x)$ is a convex function of $x$. Recall that $y(x)$ is chosen so that an incumbent politician in state $x_t$ is indifferent when choosing different values of $x_{t+1}$. An increase in $x_{t+1}$ generates a constant marginal benefit $K$, and an increasing marginal cost $e'(x_{+1})$, for the policy maker. In order to make the policy maker at $t$ indifferent between different levels of $x_{t+1}$, these effects must be compensated in equilibrium. This is achieved by a strategy where future policy makers react to the increase in $x_{t+1}$ by reducing the marginal rate of increase in the state, which eventually leads to a negative rate. In the pollution example, future policy makers move from strategic complements for low states (when they respond to increases in pollution with increases), to strategic substitutes for high states (when they respond to increase in pollution with reductions). In the equilibria of Proposition 1 this induces hump-shaped investment functions in which the rate of investment $y'(x)$ is declining, first positive and then negative as in Figure 1.

A hump shaped investment function, however, is not sufficient for an equilibrium to exist. Two other characteristics need to be satisfied, both guaranteed by the sufficient condition $R > R(\alpha)$. First, the investment function $y(x)$ must be a self map in an interval $X$ of states: once the state enters the set $X$, it must never exit.[16] Second, it must be that in $X$ there is no stable steady state: if $y(x)$ intersects the $45^o$ line, then the slope of $y(x)$ must be larger than 1 in absolute value. Condition $R > R(\alpha)$ is sufficient for these properties, and is easily satisfied. For example, we have:

$$R(1) = -\frac{2\left[1-\delta\left(1-\gamma\right)\right]\gamma}{2\delta\left(\widehat{x}\gamma+l\right)} < 0,$$

so, for any $R, l > 0$, $\delta, \gamma \in (0,1)$ and any $\widehat{x} \geq 0$, it is always the case that $R > R(\alpha)$ when $\alpha$ is sufficiently small.

[16] For example, in the equilibria of Figure 1, $y(x)$ is a self map in $\left[x_{-}^{*}, x_{+}^{*}\right]$.

A number of parameters in the model contribute to making it easier or more difficult to have equilibria with a non converging orbit. For example, it is clear that if both $\gamma = 0$ and $l = 0$, then it is impossible to construct cycles or non converging orbits. The reason is that in this case the policy constraint $x^{t+1} \geq (1-\gamma)x^t - l$ forces the policy to be monotonically increasing over time, since $x^{t+1} = y(x^t) \geq x^t$ for $\gamma = l = 0$. And indeed, consistently with this observation, we have that $R^*(\alpha) \to \infty$ as both $\gamma \to 0$ and $l \to 0$. Remarkably, however, cycles and non-converging orbits exist even for arbitrarily small $\gamma$ and $l$ if we choose the other parameters $\delta, \widehat{x}$ and $\alpha$ appropriately. Two other important variables are the discount factor $\delta$ and the ideal point for society $\widehat{x}$. The threshold $R^*(\alpha)$ is increasing in $\delta$, so the smaller is the discount factor the easier is to satisfy the sufficient condition in Proposition 1. It is indeed interesting to note that a small enough discount factor is sufficient for the existence of non converging equilibria. A small discount factor, however is not necessary, and the sufficient condition can be satisfied for any $\delta$. On the contrary, the threshold $R^*(\alpha)$ is decreasing in $\widehat{x}$, so a larger ideal point makes non-converging equilibria easier to achieve. Non-converging orbits are however possible even if $\widehat{x} \leq 0$.

The most interesting variable in $R^*(\alpha)$ is $\alpha$, which measures the degree of time inconsistency in the economy. We postpone the discussion of the relationship between time consistency and non-converging equilibria to Section 5.1.

## 3.2 Characterization of the dynamics

Proposition 1 does not specify whether the equilibrium dynamics is cyclical; and, if it is cyclical, the period of the orbit. When the equilibrium orbit converges to a stable cycle, the equilibrium is inefficient, but it is predictable since the orbit follows a well defined deterministic path. Unpredictability becomes a problem only when the orbit is aperiodic and chaotic (as defined in Section 2). In this case, the equilibrium is unpredictable because even an arbitrarily small error in measuring the state at time $t$ implies a significant error in predicting the state at $t+T$, to the extent that observing the state at $t$ may be irrelevant. We study these questions in this section: what kind of dynamics can we generate as we vary $c$ in the set $\mathcal{C}^*$?

To address this question, it is useful to "re-scale" (10) by an homeomorphism $h$.[17] Let us denote the composition of two functions by $f \circ g(x) = f(g(x))$.

**Definition 1.** *Let $f : Z_1 \to Z_1$ and $g : Z_2 \to Z_2$ be two maps, we say that $f$ and $g$ are topologically conjugate if there exist a homeomorphism $h : Z_1 \to Z_2$ such that $h \circ f = g \circ h$.*

It is important to establish whether two functions $f$ and $g$ are topologically conjugate, because topologically conjugate functions have the same dynamical properties. We have that $[f]^n = \left[h^{-1} \circ g \circ h\right]^n = h^{-1} \circ g^n \circ h$, so if $x$ is a fixed point of $[f]^n$, then $h(x)$ must be a fixed point of $[g]^n$, since we have $[g]^n \circ h(x) = h \circ [f]^n(x) = h(x)$. Indeed, the function $h$ gives a one-to-one correspondence between the periodic points of $f$ and $g$. Periodic and aperiodic for $f$ apply to qualitatively similar orbits for $g$ via homeomorphism $h$; $f$ is topologically chaotic (following Devaney [1989], Ch. 1.7.) and admits an absolutely continuous ergodic distribution if and only if the same is true for $g$. We can therefore study the properties of $f$ by studying $g$.

An adequate re-scaling of (10) by an homeomorphism simplifies the analysis of the equilibria of Proposition 1 because it allows us to link equilibrium dynamics to the dynamics of the logistic function $L_\eta(x) = \eta x(1-x)$, one of the few nonlinear functions for which the dynamics has been extensively studied (see for instance Ulam and von Neumann [1947], Ruelle [1977], Jakobson [1981], among others). Naturally, an equilibrium will never be conjugate to the logistic $L_\eta(x)$ on the entire real line, since $L_\eta(x)$ is an unbounded function while the equilibrium must satisfy the feasibility constraint $y \geq (1-\gamma)x - l$. To characterize the equilibrium dynamics, however, it is sufficient that we have conjugacy on a superset of the support of the states reached in equilibrium. We say that an equilibrium $y(x; c)$ is *topologically conjugate on its support* to $L_\eta$ if there is a set

[17] A function between two topological spaces $I$ and $J$, $g : I \to J$, is said to be a *homeomorphism* if it is one-to-one, onto, continuous, and its inverse $g^{-1}$ is also continuous.

$I$ such that $[y]^m(x;c) \in I$ for all $m \geq \underline{m}$ for some finite $\underline{m}$ and $x \in I$, and $y(x;c)$ is topologically conjugate to $L_\eta$ on $I$. We have:

**Lemma 1.** *Assume $R \geq R^*(\alpha)$ as defined in Proposition 1. For any $\eta \in [3,4]$, there is a constant:*

$$c(\eta;\varphi_1,\varphi_2) = \frac{1}{\varphi_2}\left[(\varphi_1/2)(1-\varphi_1/2) - \eta/2(1-\eta/2)\right] \tag{13}$$

*such that $c(\eta;\varphi_1,\varphi_2) \in \mathcal{C}^*$ as defined in (12), and $y^*(x;c(\eta;\varphi_1,\varphi_2))$ is an equilibrium that is topologically conjugate to $L_\eta$ on $X^*(c(\eta;\varphi_1,\varphi_2))$.*

Lemma 1 allows to construct equilibria with cycles of various periods. It is well known that for values $\eta \in \left[3, 1+\sqrt{6}\right]$, the logistic has a unique stable cycle of period 2.[18] We can therefore construct an equilibrium with a cycle of period 2 by setting the constant $c^*$ in the equilibrium $y^*(x,c^*)$ of Proposition 1 at $c^* = c(\widehat{\eta};\varphi_1,\varphi_2)$ for any $\widehat{\eta} \in \left[3, 1+\sqrt{6}\right]$. For example, the equilibrium in the top panel of Figure 1 is constructed by setting $c^*$ in (10) equal to $c(7/2;\varphi_1,\varphi_2)$. As $\eta$ increases beyond $1+\sqrt{6}$, cycles of order $2m$ for any $m \geq 1$ emerge; and for $\eta > \eta_\infty = 3.5699$ there are isolated values of $\eta$ for which cycles with odd periods appear (as the value of the example in Figure 1 where $m = 3$).[19] The equilibrium with a stable cycle of period 3 in Figure 1 is indeed constructed setting $c^* = c(3.835;\varphi_1,\varphi_2)$.[20]

In addition to stable cycles, the literature has also identified specific values of $\eta$ in $[\eta_\infty, 4]$ for which $L_\eta(x)$ displays chaotic behavior, for example $\eta = 4$ (see Ulam and von Neumann [1947]); or the Ruelle's constant $\eta^*$, which is approximately 3.6785735 (see Ruelle [1977]).[21] Lemma

[18] See Devaney (1989), among others.

[19] The existence of stable cycles of order 3 is particularly important because by the Sarkovski Theorem they imply the existence of cycles of any order. This has sometimes been equated to the presence of chaos. This is however not a completely legitimate interpretation. The logistic has a unique stable cycle and the dynamics converges to it starting from all points in its support except from a subset of measure zero. The additional cycles are unstable cycles that exist only for initial values in a set of measure zero. These cycles are often referred to as "invisible" since for all practical purposes they are unobserved.

[20] For the fact that the logistic with constant equal to 3.835 generates a stable cycle of period 3 (and for specific intervals for which the logistic has stable cycles of higher periods) see, for example, May and Oster [1976].

[21] Ruelle's constant $\eta^*$ is the the only real solution $\eta^*$ of $(\eta^*-2)^2(\eta^*+2) = 16$.

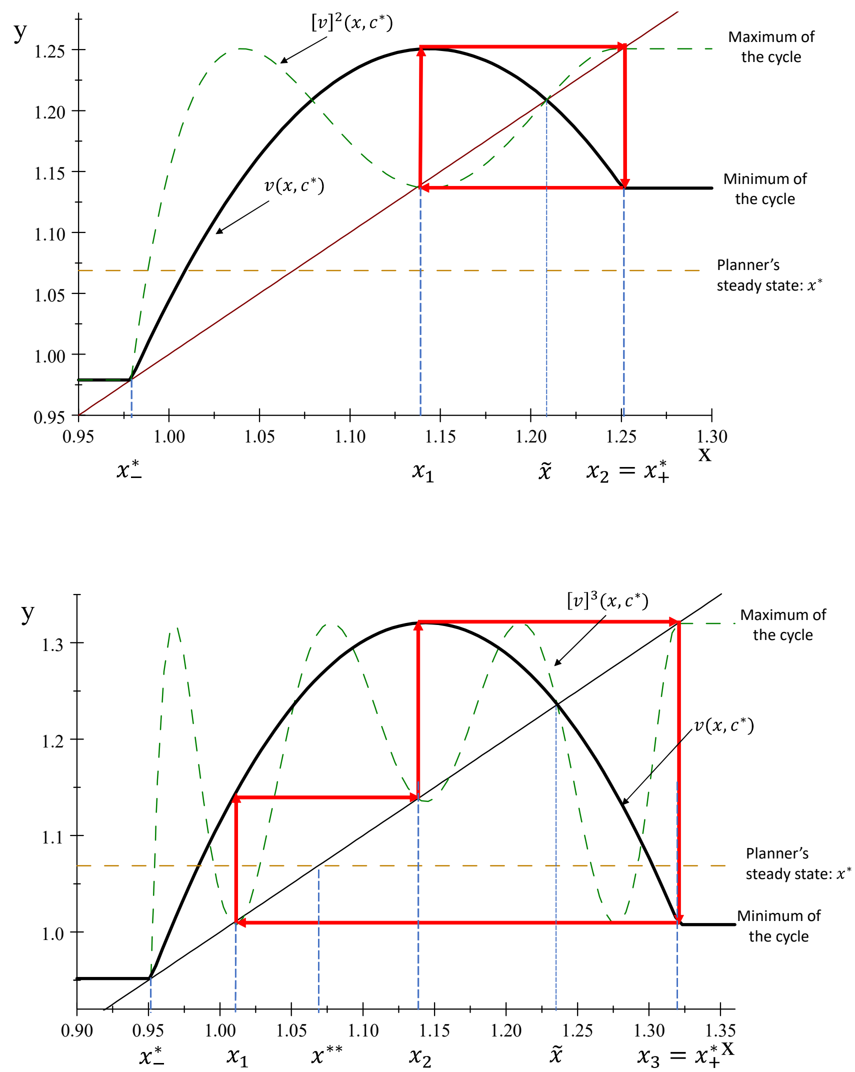


Figure 1: One economy, two equilibria with stable cycles of periods $m = 2$ and $m = 3$. The solid (black) line are the investment function $y(x)$, the dashed (green) line are the iterated maps $[y]^2(x)$ and $[y]^3(x)$. Note that the bounds $x^*_-$, $x^*_+$ defining $X^*(c^*) = [x^*_-, x^*_+]$ are different in the two equilibria.

1 shows that both $c(4;\varphi_1,\varphi_2)$ or $c(\eta^*;\varphi_1,\varphi_2)$ are in $\mathcal{C}^*$: we can therefore generate equilibria with the same qualitative properties setting $c = c(4;\varphi_1,\varphi_2)$ or $c(\eta^*;\varphi_1,\varphi_2)$. The top panel of Figure 2 presents the orbit of two chaotic equilibria with $c$ equal to $c(4;\varphi_1,\varphi_2)$ and $c(\eta^*;\varphi_1,\varphi_2)$, respectively.

It is important to note that the conditions under which the logistic has a cycle of period 3 is non generic: we obtain a cycle 3 with $\eta = 3.835$, but if we perturb this parameters, we do not necessarily obtain a cycle of period 3. Similarly, the constant $c_3 = c(3.835;\varphi_1,\varphi_2)$ that induces an equilibrium with cycle 3 in an economy with parameters $\varphi_1,\varphi_2$ is not robust; if we perturb the economy to a nearby parametrization $\widetilde{\varphi}_1,\widetilde{\varphi}_2$ then we do not attain a cycle of period 3.

The important observation, however, is that the existence of the equilibrium is indeed a generic phenomenon. An equilibrium with cycle 3 continues to exist if we perturb to a nearby $(\widetilde{\varphi}_1,\widetilde{\varphi}_2)$, yet the constant would move continuously to $c_3 = c(3.835;\widetilde{\varphi}_1,\widetilde{\varphi}_2)$. The constant $c$ in the equilibrium construction is an *endogenous* variable, not a parameter given by nature as $\varphi_1,\varphi_2$. Proposition 1 and Lemma 1 prove that, for *any* generic economy satisfying $R \geq R^*(\alpha)$, there exists an equilibrium with these properties. Furthermore, the value $c(\eta;\varphi_1,\varphi_2)$ that achieves these properties is not assumed in the environment, and is endogenously determined in equilibrium. Summarizing these considerations, we have the following result.

**Proposition 2.** *Assume an economy with $R \geq R^*(\alpha)$ as defined in (11):*

- *For every value $m \geq 2$, there is an equilibrium $y^*(x,c_m^*)$ associated to a point $c_m^* \in \mathcal{C}^*$ with a unique stable cycle of period at least $m$. The orbit of this equilibrium is in $X^*(c_m^*)$ and characterized by the fixed points of $[y]^m(x;c_m^*)$, as defined in (10).*

- *There is also a non empty subset of $\mathcal{C}^D \subset \mathcal{C}^*$ such that the associated equilibria $y^*(x,c^*)$ with $c^* \in \mathcal{C}^D$ display topological chaos a' la Devaney on $X^*(c^*)$.*

Given Proposition 2, it is also natural to ask what the properties of the long term distribution of states induced by iterations of $y$ are. A particularly important property is whether the distribution is absolutely continuous, invariant and ergodic.[22]

**Definition 2.** *We say that a dynamical system displays ergodic chaos if there is an absolutely continuous probability measure that is ergodic and invariant.*

When we have ergodic chaos, the behavior of the dynamical system in the long term is completely independent of the starting point, and it can be described by a distribution function. Ulam and von Neumann [1947] have famously shown that $L_4(x)$ admits an ergodic distribution that can be characterized in closed form and is equal to the arcsine distribution with density: $\lambda(x) = \pi^{-1}(x(1-x))^{-1/2}$. While this is the only case for which the ergodic distribution of the logistic has been characterized in closed form (and one of the very few dynamical systems for which it can be characterized), subsequent work has shown that there is a set of positive measure of values of $\eta$ such that $L_\eta(x)$ admits an ergodic distribution (one of which is Ruelle's number $\eta^*$). Jacobson [1981] and Benedicks and Carleson [1985], moreover, have shown that is a positive measure of $\eta \in [0,4]$ for which $L_\eta(x)$ admits an ergodic distribution and a positive Lyapunov exponent, which means that neighboring orbits separate exponentially fast (one of them is indeed $\eta = 4$).

By Lemma 1, for each of these values, there is an equilibrium of the policy game that qualitatively inherits the same properties. The case with $c(4; \varphi_1, \varphi_2)$ inherits the qualitative properties of Ulam and von Neumann's case, though in a space translated by a specific homeomorphism. This implies that the distribution generated by the equilibrium correspondent to $c(4; \varphi_1, \varphi_2)$ can be characterized in closed form, though now its "shape" depends on the fundamentals of the economy. Define the following density $\mu(x;\omega)$ on $X^*$:

[22] A distribution $\mu$ is said to be invariant if $y_*\mu = \mu$, where $y_*\mu$ is the push forward measure $y_*\mu(A) = \mu(y^{-1}(A))$. A distribution is ergodic if $\lim_{T\to\infty} \sum_{k=1}^{T} \varphi([y]^k(x)) = \int \varphi d\mu$ for almost all $x$.

$$\mu(x;\omega) = \frac{2R}{\pi}\left(16\left(1-\alpha\right)^2 - \left(2Rx - \left[\begin{array}{c}\frac{2}{\delta} - (1+\alpha)(1-\gamma) \\ +2R\cdot\widehat{x}\end{array}\right]\right)^2\right)^{-1/2}. \tag{14}$$

where $\omega = (R, \alpha, \delta, \gamma, \widehat{x})$ is the vector of parameters characterizing the economy. We have:

**Proposition 3.** *Assume an economy with $R \geq R^*(\alpha)$. There is a subset of $\mathcal{C}^E \subset \mathcal{C}^*$ with positive measure such that the equilibria $y(x, c^*)$ with $c^* \in \mathcal{C}^E$ display ergodic chaos on the compact on support $X^*(c^*)$. Among these equilibria, the equilibrium $y^*(x, c(4; \varphi_1, \varphi_2))$ admits the invariant distribution $\mu(x;\omega)$ on $X^*(c(4; \varphi_1, \varphi_2))$ defined above.*

An equilibrium constant $c(\eta^*; \varphi_1, \varphi_2)$ that works to generate a chaotic equilibrium for an economy $\varphi_1, \varphi_2$ does not work for a perturbed economy $\widetilde{\varphi}_1, \widetilde{\varphi}_2$. As for the results of Proposition 2, however, the existence result presented in Proposition 3 holds for a generic choice of the parameters $\varphi_1, \varphi_2$ defining the economy, not just for specific non generic values: if we perturb the economy, then there exists a constant $c(\eta^*; \widetilde{\varphi}_1, \widetilde{\varphi}_2)$ in correspondence of which the equilibrium is chaotic for the perturbed economy. While the fact that we have topological chaos in the logistic with specific values $\eta = 4$ or $\eta^*$ may appear as a mathematical curiosity, in Proposition 3 we show that topological and ergodic chaos exists for very large, non-generic economies.

To conclude this discussion, we note the geometric properties characterizing the investment functions $y(x)$ that generate ergodic chaos. First note that all equilibrium investment functions $y(x)$ of Proposition 1, both with and without stable cycles, qualitatively look like the hump shaped functions illustrated in Figure 1. While there is no general characterization of necessary and sufficient conditions on the shape of the transition function $y$ to generate ergodic (or topological) chaos, there is a known general sufficient condition that gives us insights on the geometric features of $y$ that are associated to chaotic behavior.[23] Let $x^*$ be the critical point of $y(x)$, the point that maximizes $y(x)$; let $x^*_-$ and $\widetilde{x}$ be the lower and higher fixed points of $y(x)$, respectively. For

[23] The examples presented in Figure 2 satisfy this condition.

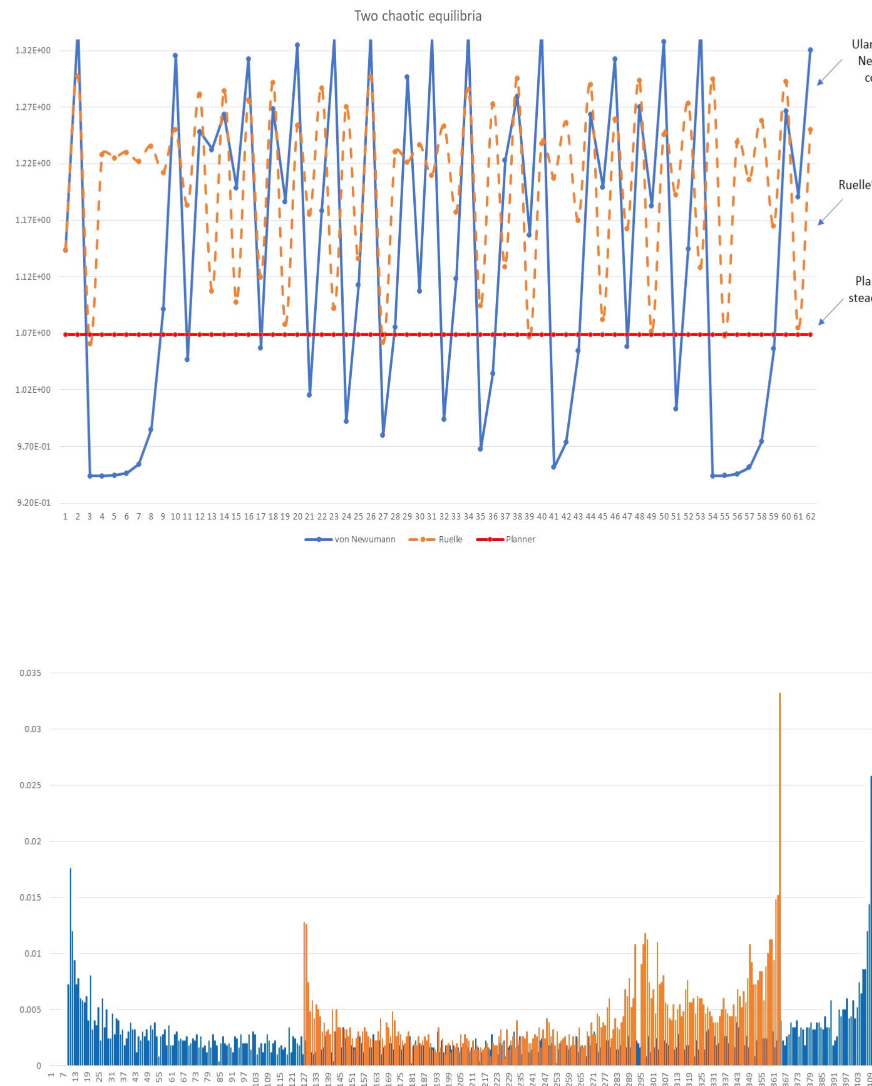


Figure 2: Two equilibria with ergodic chaos existing for the same economy. The first equilibrium is topologically conjugate to von Neumann and Ulam's example, the second to Ruelle's example. The top panel displays the state trajectories; the second the ergodic distributions.

these investment functions, $x^* \in (x^*_-, \widetilde{x})$, so the maximum $y(x^*)$ is on the left of the $45^o$ line: for any initial $x_0 \in [x^*_-, f(x^*)]$, the state remains in $X = [x^*_-, f(x^*)]$ for all iterations. Moreover, we have no stable steady state because the slope of $y$ is larger in absolute value than one at $x^*_-$ and $\widetilde{x}$ (the two unstable steady states). Given this, there are three forces "pushing around"the state. When the state is close to $\widetilde{x}$, it is repelled by it since $y'(x) < 1$. The state can move down below $\widetilde{x}$, or up above $\widetilde{x}$. If the state is pushed down toward $x^*_-$, then it is repelled again to a higher state, so the state must be eventually be pushed up, above $\widetilde{x}$. In this case, however, it eventually reaches a point $x < f(x^*)$ at which $f(x) < x$, so it will have to move down. These dynamics may induce the system to converge to a cycle with support in $(x^*_-, f(x^*))$, as in Figure 1. It can however be shown that under regularity conditions satisfied by the equilibria of Proposition 1, a necessary condition for this to occur (and thus for the existence of a stable cycle) is that the orbit originated from the critical point $x^*$ converges to a stable cycle (see Theorem II.4.1 in Collet and Eckmann [1980]). It follows that whenever the orbit starting from the critical point does not converge to a stable cycle, then a stable cycle of any period does not exist. While this does not necessarily imply that we have ergodic chaos, it is indeed the case with sufficient regularity – as the equilibria of Proposition 1.[24]

# 4 General preferences

In the previous analysis, we made two simplifying assumptions on the functional form of the utility function. In (3), we assumed a quadratic cost function $e(x)$ and, more importantly, quasi-linear preferences. We discuss and relax these assumptions in this section.

The assumption of a quadratic $e(x)$ allows us to provide a closed form solution for $y(\cdot, c)$ in (10) and an exhaustive characterization of the possible equilibrium dynamics. Dispensing this

[24] Ergodic chaos requires convergence of the ergodic distribution to an absolutely continuous distribution. A "geometric"sufficient condition is presented by Misiurewicz [1981]; it requires that the orbit starting from the critical point $x^*$ enters an unstable cycle. This condition is satisfied by both examples in Figure 2.

assumption, however, does not qualitatively change the analysis. The differential equation (9) does not require a quadratic functional form and, for a generic convex differentiable $e(x)$, it generates a non-monotonic, hump shaped investment function as (10), a feature that is key to obtain cycles and complex dynamics (see Figure 1 for an illustration). Whether such an investment function generates a stable steady state, cycles or a chaotic trajectory depends on the curvature of $e(x)$. In Section 8.1 of Appendix B we present a general sufficient condition for the existence of chaotic equilibrium trajectories for the $y(\cdot, c)$ that solves (9), and we apply it in various examples under various standard cost functions.

Relaxing the assumption of quasi-linear preferences has a more interesting impact on the analysis because, although the logic remains the same, it requires a generalization of the equilibrium construction. We discuss it in the reminder of this section. Consider the more general utility functions described in Section 2 and define for convenience:

$$u\left(y,x\right)=u^{i,i}\left(y-(1-\gamma)x,x\right), \text{ and } \upsilon\left(y,x\right)=u^{i,j}\left(y-\left(1-\gamma\right)x,x\right) \tag{15}$$

for $i$ and $j \neq i$ to be, respectively, the utility of a citizen when their party is in office, and when their party is out of office. To study this environment it is useful to move beyond the special case of Markov equilibria and define a slightly more general strategy function $y(z, x)$ with one period memory, depending on the current state $z$ and the precedent state $x$. The strategy defined in the previous section can be seen as a special case of $y(z, x)$.

The problem of the incumbent at $t-1$ in state $x_{t-1} = x$ is to find a state $z$ that solves:

$$\max_{z\geq(1-\gamma)x-l}\left\{u\left(z,x\right)+\delta v(z,x)\right\}. \tag{16}$$

where $v(z, x)$ is the expected value function at $t$ of a policy maker when the state is $x_t = z$ and $x_{t-1} = x$ (the state $x_{t-2}$ is irrelevant for the set of solutions of (16), so it can be ignored for the discussion here). Similarly as in the analysis of Section 3.1, $v(z,x)$ can be written as:

$$\begin{aligned} v(z,x) &= \frac{1}{2}u\left(y(z,x),z\right)+\frac{1}{2}\upsilon\left(y(z,x),z\right)+\delta v(y(z,x),z) \\ &= u\left(y(z,x),z\right)+\delta v(y(z,x),z)-\Phi\left(y(z,x),z\right) \end{aligned} \tag{17}$$

where we define $\Phi\left(y(z,x),z\right)=\frac{1}{2}\left[u(y(z,x),z)-\upsilon(y(z,x),z)\right]$. Assuming differentiability without loss of generality, the envelope theorem allows us to differentiate the value function with respect to the first argument, $z$:

$$v_1(z,x)=u_2(y(z,x),z)-\Phi_1(y(z,x),z)y_1(z,x)-\Phi_2(y(z,x),z)+\delta v_2(y(z,x),z) \tag{18}$$

where $v_l$, $u_l$ and $\Phi_l$ are the derivatives with respect to the $l$th arguments of $v$, $u$ and $\Phi$. This condition still depends on the value function, through its derivative with respect to the second term. Using (17) again, we can see that this derivative in a state $s,z$ can be written as:

$$\begin{aligned} v_2(s,z) &= \left[u_1(y(s,z),s)+\delta v_1(y(s,z),s)\right]y_2(s,z)-\Phi_1(y(s,z),s)\cdot y_2(s,z) \\ &= -\Phi_1(y(s,z),s)\cdot y_2(s,z) \end{aligned} \tag{19}$$

where in the last equality we again apply the envelope theorem. Combining (18) and (19), and using the first order necessary condition from (16), we obtain:

$$\begin{aligned} -\frac{u_1(z,x)}{\delta} &= u_2(y(z,x),z)-\Phi_1(y(z,x),z)y_1(z,x)-\Phi_2(y(z,x),z) \\ &\quad -\delta\Phi_1(y(y(z,x),z),y(z,x))\cdot y_2(y(z,x),z) \end{aligned} \tag{20}$$

Condition (20) and the function $y(z,x)$ that satisfies it play the same role as condition (9) and $y(\cdot)$ studied in Section 3.1. The key difference is that while (9) defined a simple differential equation, now the functional equation (20) defines a significantly more complex partial differential equation (PDE). The reason for this is intuitive. As the policy maker in $x$ selects $z$ at $t-1$, a marginal change affects the future in two ways. First, it affects the policy $y(z,x)$ chosen at $t$, which is a function of $z$ as in Section 3. The policy change at $t$ affects the expected value function at $t$ because the envelope theorem does not fully apply given that the problem is time inconsistent, as discussed in the previous section. But now we have a novel second effect. A marginal change of the policy $z$ at $t-1$ also affects the way in which the policy maker at $t+1$ reacts to the policy maker at $t$: i.e., for any choice $y(z,x)$ made at $t$, now a marginal change in $z$ induces a policy change at $t+1$ by $y_2(y(z,x),z)$. Once again, part of this effect is "neutralized" by the envelope theorem applied at $t+1$; but part of it remains through the marginal effect on $\Phi(y(y(z,x),z),y(z,x))$.

The second effect described above is present because we have assumed a strategy $y(z,x)$ with one period memory. Condition (20) makes it clear why it is necessary to consider a strategy $y(z,x)$ with one period memory. When $u(z,x)$ is quasi-linear as in the previous sections, $u_1(z,x)$ is independent of $x$. It follows that the right hand side of (20) is also independent of $x$, implying that the equilibrium strategy satisfying (20) must be only a function of $z$. In the general case, however, $u_1(z,x)$ is a function of $x$, implying that we need to allow $y(z,x)$ to be a function of $x$ as well to satisfy (20). The reason is that $y(z,x)$ is designed to make the policy maker in state $x$ indifferent with respect to $z$: if the marginal utility of a change in $z$ at $t-1$ depends on $x$, then future policy makers must adjust accordingly.

The functional equation (20) combines elements of a PDE, since it is a function of the partial derivatives $y_1(z,x)$ and $y_2(y(z,x),z)$; and elements of a difference equation, since the partial derivatives are evaluated at states $x_{t-1}=x$, $x_t=z$ and states $x_t=z$, $x_{t+1}=y(z,x)$. Because of this (20) does not allow to have a simple analytical characterization, except for the quasi linear case studied in the previous section (which can be seen as just a special case). Still, the analysis of the previous section helps because it gives us a closed form solution in the limit case with quasi linear utilities that can be used compute solutions to (20) numerically.

Consider the family of utilities $u^{i,j}(p,x;\xi)$, parametrized by $\xi$ such that $u^{i,j}(p,z;\xi)$ is contin-

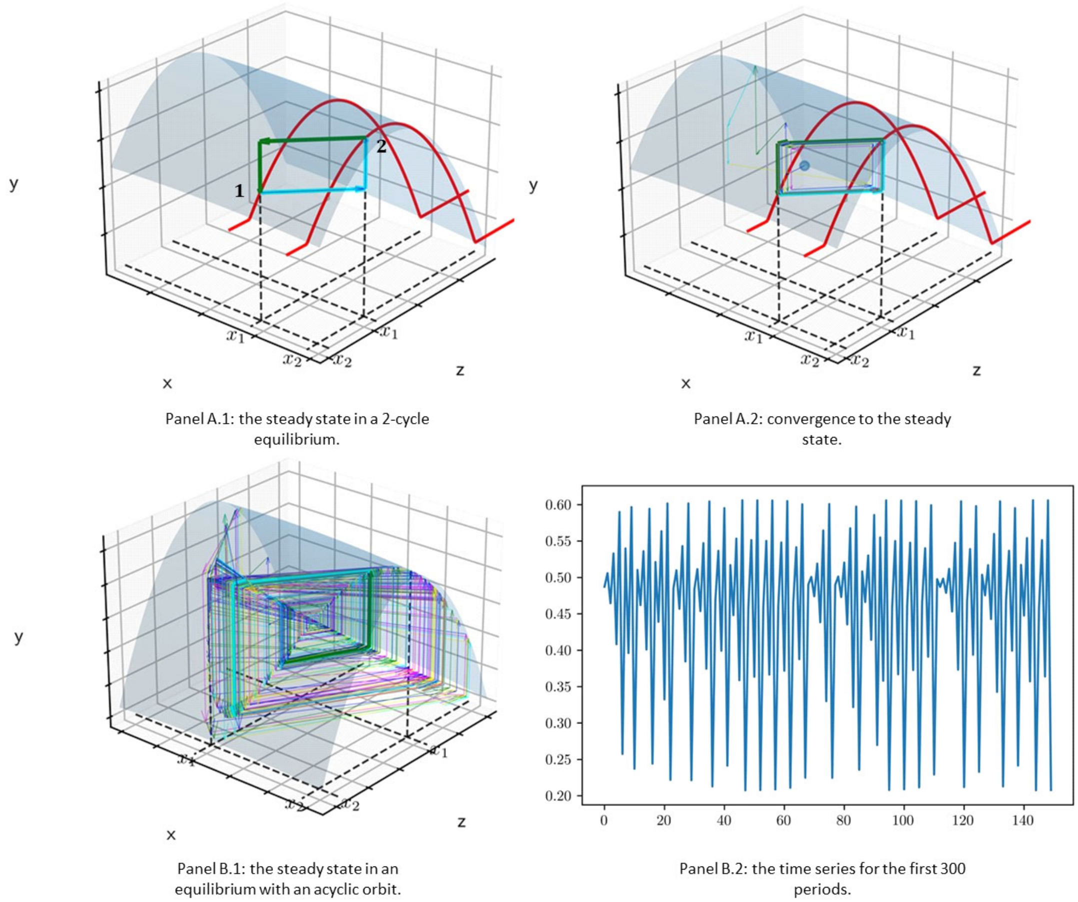


Figure 3: An equilibrium with a 2-period cycle (Panels A1 and A2), and an equilibrium with a aperiodic orbit (Panels B1 and B2), both with preferences (21).

uously differentiable in all arguments. A simple example from this family is:

$$u^{i,j}(p, x; \xi) = \alpha_{i,j} K \cdot (\theta_1 + \theta_2 p)^{1-\xi} / (1 - \xi) - e(x), \tag{21}$$

where $p = y - (1-\gamma)x$ and $\xi$, $\theta_1$, $\theta_2$ are positive parameters.[25] The parameter $\xi \in [0, 1]$ measures the elasticity of the utility of $\theta_1 + \theta_2 p$: we have the quasi linear utility of the previous sections when $\xi = 0$. The quasi-linear preferences of Section 3 are weakly supermodular, in the sense that

[25] We allow here for a constant $\theta_1$ to permit environments in which $p$ can be zero or negative, values in correspondence of which the utility is either not defined, or not a real number.

the cross derivative of the per period utility (3) with respect to $x$ and $y$ is zero. The utility in (21) is non-linear in $p$ and is strictly supermodularity with respect to $x$ and $y$ for $\xi > 0$.

Using the solution for $\xi = 0$ as starting condition, we can solve for the equilibria when $\xi$ is strictly positive, but small. Panels A.1 and A.2 in Figure 3 illustrate an example in which the equilibrium orbit converges to a 2-priod cycle with periodic points $x_1$ and $x_2$, assuming (21) and the elasticity parameter $\xi$ equal to 5%. The shaded surfaces in Panels A.1-A.2 describe the solution of (20) that characterizes the equilibrium.[26] Panel A.1 describes the steady state. Consider a state $(x^2, x^1)$ in the cycle, corresponding to $x_t = x^2$, $x_{t-1} = x^1$. The policy maker selects point 1 with $x_{t+1} = y(x^2, x^1) = x^1$ lying on the curve $y(z; x^1)$, illustrated by the solid red curve. This moves the state to $\left(x^1, x^2\right)$ at $t + 1$. In $\left(x^1, x^2\right)$, the choice is $x_{t+2} = y(x^1, x^2) = x^2$ (i.e. point 2 in Panel A.1), moving the state back to $(x^2, x^1)$ at $t + 2$. Panel A.2 of Figure 3 illustrates the same equilibrium, but highlighting the convergence path starting from a point outside the steady state cycle.

Equilibria with seemingly chaotic behavior can also be constructed. Panels B.1 and B.2 in Figure 3 illustrate in the state space such an equilibrium for the same parametrization as in the examples of Panels A.1-A.2. (see Panel B.1), and the associated time series described by the orbit (panel B.2). We qualify the dynamics in Panels B.1 and B.2 as "seemingly" chaotic because, differently from the results of Section 3, we do not have a formal proof that the orbit illustrated is chaotic in the formal sense described in Section 2-3. The numerical solution, however, shows that at least for the first 200 periods illustrated in Panel B.2, the orbit is aperiodic and extremely complex.

[26] A detailed description of the algorithm used to solve (20) and compute the equilibria presented in Figure 3 is presented in Section 8.2 of Appendix B. Specifically, in the computations of Figure 3 we assume $e(x) = (\beta/2)\left(x - \widehat{x}\right)^2$, with $\theta_1 = 2$, $\theta_1 = 1$, $K = 0.4$, $\beta = .75$, $\gamma = .07$, $\widehat{x} = .15$ and $\delta = .85$.

# 5 The role of time inconsistency

## 5.1 Time inconsistency and the "size" of the chaotic region

In the model of Sections 2-4, the degree of time inconsistency impressed on the economy by the decision process plays a particularly important role in determining "how much" chaos we can observe in equilibrium. Time inconsistency consists in the discrepancy between the value of the objective function maximized by the incumbent selecting the policy (i.e. (5)), as well as in the expected continuation value at $t$ before the incumbent at $t+1$ is determined (i.e. (7)). In the planner's problem there is no difference between these two functions. By comparing (5) and (7), we can see that in the political game, instead, the functions differ by:

$$-\frac{(1-\alpha)K}{2}\left[y(x)-(1-\gamma)x\right].$$

The parameter $\alpha \in (-\infty, 1]$ captures time inconsistency: as $\alpha \to 1$, time inconsistence converges to zero; as we reduce $\alpha$, time inconsistency is increased.[27] In this limit case as $\alpha \to 1$, $v'(x)$ is independent of $y'(x)$ and the equilibrium qualitatively looks like the planner's problem. In this case the optimal policy for the incumbent ignoring the feasibility constraint is $y_{nt} = \widehat{x} + \frac{K}{2\beta} \cdot (1/\delta - (1-\gamma))$, and so the equilibrium policy is $y_{nt}(x) = \max\{y_{nt}, (1-\gamma)x - l\}$. This simple dynamical system has a unique steady state $y_{nt}$ that differs from the planner's steady state $Y^*$ only because it is lower. The effect of $\alpha$ on the "degree of chaos" in the equilibria constructed in the previous sections can be seen from the size of the set $X^*(c)$ which contains (except at most for a finite transition period) the trajectory: $\|X^*(c)\| = \overline{x}(c) - \underline{x}(c)$, where $\underline{x}(c)$ and $\overline{x}(c)$ are the bounds defined in (12). Using the bounds on $c$ in $\mathcal{C}^*$, it is easy to see that $\|X^*(c)\| \leq 4(1-\alpha)K/\beta \to 0$ as $\alpha \to 1$. It follows that as $\alpha \to 1$ we can still have chaos, but the size of the set in which the state can "wander around" collapses to zero.

[27] When $\alpha = 1$, the effect of private expenditure is the same on the constituencies of the party in power and of the other party. The equilibrium policy is still not the utilitarian policy because the incumbent does not internalize the negative externality of $x$; but it is time consistent.

## 5.2 Alternative economic models with time inconsistency

In the previous analysis we have assumed a stylized dynamic political economy model in which two parties alternate in power. In this section, we show that the logic behind the type of equilibria with cycles and chaotic behavior studied above applies to a variety of other important economic problems. In the following we briefly illustrate how to the analysis can be applied to three distinct examples: a single agent decision making problem with hyperbolic discounting; a dynamic multi-agent problems with free riding; and a case in which policies are decided in a process of non-cooperative bargaining as in Battaglini and Coate [2007, 2008] and Battaglini [2011].

### 5.2.1 Hyperbolic discounting

In the case of a single decision maker with $\beta$-$\delta$ preferences as in Phelps and Pollak [1968] and Laibson [1997], the policy solves:[28]

$$\max_{y \geq (1-\gamma)x-l} \left\{ K\left[y-(1-\gamma)x\right]-e(x)+\beta\delta v(y) \right\}, \tag{22}$$

where the only difference with (5) is that there is an additional term, the hyperbolic discount factor $\beta < 1$. The decision maker plays a game against his/her future selves. The expected continuation $v(x)$, must satisfy: $v(x) = K(y(x)-(1-\gamma)x)-e(x)+\delta v(y(x))$, where $y(x)$ is the expected future policy and $\beta$ does not appear. This expression can be written as:

$$v(x) = \max_{y \in [(1-\gamma)x-l, \overline{x}]} \left\{ K\left[y-(1-\gamma)x\right]-e(x)+\beta\delta v(y) \right\} + (1-\beta)\,\delta v(y(x)). \tag{23}$$

Condition (22) and (23) corresponds to conditions (5) and (7) presented above. The second term in (23), $(1-\beta)\,\delta v(y(x))$, is the time inconsistency gap: i.e. the difference between the decision-maker's objective function and the expected value function. Because of this additional term, the shape of the expected value function directly depends on the expected future investment function

[28] The parameter $\beta$ was used in Sections 3 and 4 in the definition of $e(x)$. We use it it for the time preferences since this is the traditional notation for hyperbolic discounting and there is no risk of confusion since $e(\cdot)$ is can be assumed here to be a general convex function.

$y(x)$ as in (7). Differentiating (23) and using the first order necessary condition $K/(\beta\delta) = -v'(x)$ from (22), we obtain a condition analogous to (10) in the analysis of Section 3: $y(x) = \left[\frac{1-(1-\gamma)\delta\beta}{\delta(1-\beta)}\right] \cdot x - \frac{\beta}{(1-\beta)k} \cdot e(x) + c$.

#### 5.2.2 Dynamic free riding

Battaglini et al. [2014] propose a dynamic model of free riding with $n$ agent in which a public good is accumulated by independent voluntary contributions. Each agent is endowed with a per period budget of $W/n$ to allocate in each period $t$, either to private consumption or to increase the stock of the public good $y$. Battaglini et al. [2014] shows that, in state $x$, each player selects the individual contributions as if they could directly choose the new state $y$ to maximize $u(y) + \left[\frac{W+(1-\gamma)x}{n} - y\right] + \delta v(y)$ under a feasibility constraints $y \in F(x)$. As in (23), the value function diverges from each player's objective function and can be written as:

$$v(x) = \max_{y \in F(x)} \left[u(y) + \left[\frac{W + (1-\gamma)x}{n} - y\right] + \delta v(y)\right] + \left(1 - \frac{1}{n}\right) y(x),$$

where $y(x)$ is the expected equilibrium investment function. While Battaglini et al. [2014] focus on monotonic equilibria with no cycles, the fact that the second term in the value function $v(x)$ directly depends on $y(x)$ allows us to construct cycles of any order, as well as chaotic equilibria as in the previous sections in this environment.

#### 5.2.3 Noncooperative legislative bargaining

Consider a version of the model presented in the previous sections in which, in state $x$, a governmental policy consists in a durable public good $y$ that directly affects the state variable $x$ as above; and now, in addition, a vector of transfers $\mathbf{s} = (s_1, ..., s_n)$, where $s_i$ is targeted to electoral district $i$. For simplicity here we assume that there are no other taxes/transfers and that the total direct benefit from $y$, i.e. $K[y - (1-\gamma)x]$, is directly appropriated by the government. Electoral districts are assumed to be symmetric, each comprising the same mass of citizens. A policy is approved if the representatives of at least $q$ electoral districts approve it. The government redistributes $K[y-(1-\gamma)x]$ as part of the policy: so $(y,\mathbf{s})$ must satisfy the budget constraint $\sum_i s_i = K[y-(1-\gamma)x]$ in state $x$.[29]

The bargaining protocol in a period $t$ is as in Battaglini and Coate [2007, 2008] and Battaglini [2011]: at each stage $\tau$ of the protocol, a representative from electoral district $i=1,...,n$ (the proposer) is selected to propose a policy $(y,\mathbf{s})$ with probability $1/n$; the proposer aims at maximizing the utility of the citizens in his/her district. If the proposal is accepted, then the policy is implemented and the legislature adjourns, meeting again in period $t+1$ with new state $y$; if the proposal fails, a new representative is randomly selected at stage $\tau+1$ of period $t$ with probability $1/n$ to make a new proposal and the process repeats in the same state $x$.[30] We assume that the time lost between proposals is negligible.

In a symmetric equilibrium the proposer makes a transfer $s$ to $q-1$ randomly selected other representatives to win a minimal winning coalition, and zero to the representatives outside the selected minimal winning coalition. The variables $y$ and $\mathbf{s}$ are selected to maximize the proposer's expected utility: $K[y-(1-\gamma)x]-(q-1)s+\delta v(y)$. The "bribe" $s$ must satisfy the incentive constraint that guarantees that the representatives in the minimal winning coalition vote accept the proposal $y,\mathbf{s}$, so $s \geq v(y(x)) - \delta v(y)$. Given this, it can be shown that the equilibrium policy $y$ solves:

$$\max_{y \geq (1-\gamma)x-l} \left\{ \frac{K[y-(1-\gamma)x]}{q} + \delta v(y) \right\}, \tag{24}$$

On the other hand, in a symmetric equilibrium, the value function is $v(x) = \frac{K[y(x)-(1-\gamma)x]}{n} +$

[29] For a more sophisticated version of this model in which the direct benefit of pollution are appropriated by the citizens and trasfers are financed by tax revenues, see Battaglini and Coate [2007].

[30] This version of the bargaining protocol is presented in Battaglini [2011]. The timing in Battaglini and Coate [2007, 2008] is slightly different, but it generates the same equilibrium conditions. The discussion below, therefore, applies to these cases as well.

$\delta v(y(x))$, that can be written as:

$$\max_y \left\{ \frac{K[y-(1-\gamma)x]}{q} + \delta v(y) \right\} - K\left(\frac{n-q}{q}\right)\left[\frac{y(x)-(1-\gamma)x}{n}\right]. \tag{25}$$

Once again, condition (24) and (25) corresponds to conditions (5) and (6).

# 6 Conclusions

In this paper we have studied a simple dynamic game in which in every period a politically motivated decision maker selects a policy that affects a state variable strategically linking policy making periods. Because of political turnover, the preference of the policy maker may change over time, causing the decision process to be time inconsistent. We ask the question: under what conditions can such a simple model generate cycles and complex, unpredictable dynamics? Complex dynamics are impossible when policies are selected by a benevolent, time consistent policy maker. In the presence of time inconsistency generated by the political process, however, simple sufficient conditions guarantee the existence of equilibria with cycles of any order and even chaos for generic economies. The degree of instability and unpredictability depends on the degree of time inconsistency: as time inconsistency converges to zero, chaotic equilibria continue to exist, but the size of the region containing the chaotic or cycling trajectory vanishes.

A limitation of our results is that the chaotic behavior we characterize is not typical of all equilibria of our dynamic economy, but instead of the specific class of equilibria that we have characterized. Still, they show simple yet realistic environments in which predicting dynamic public policies is impossible in the sense that there are always chaotic equilibria that make it impossible. The problem is not that there are multiple equilibria, but that once we know the equilibrium, the dynamics are effectively unpredictable, even without random shocks to the system. Equilibria with complex dynamics, moreover, highlight a new source of inefficiency generated in political equilibria that has no correspondent in standard planner's problems: the instability of policies even in the absence of external shock.

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# 7 Appendix A

## 7.1 Proof of Proposition 1

Let us rewrite (10) as $y(x,\kappa) = \varphi_1 x - \varphi_2 x^2 + \kappa$, where $\varphi_1 = \frac{1}{1-\alpha}\left[\frac{2}{\delta} - (1+\alpha)(1-\gamma) + \frac{2\beta}{K}\widehat{x}\right]$ and $\varphi_2 = \beta/\left[(1-\alpha)K\right]$. For any real number $\kappa$ satisfying $\kappa \geq \left[4 - (\varphi_1 - 1)^2\right]/4\varphi_2$, define $\widehat{x}_-$, $\widehat{x}_+$ as, respectively, the lowest and the largest fixed points of $y(x,\kappa)$. We have:

$$\widehat{x}_- = \frac{\varphi_1 - 1 - \sqrt{(\varphi_1-1)^2 + 4\varphi_2\kappa}}{2\varphi_2},\ \widehat{x}_+ = \frac{\varphi_1 - 1 + \sqrt{(\varphi_1-1)^2 + 4\varphi_2\kappa}}{2\varphi_2}.$$

Define the function:

$$Y(x;\kappa,\underline{x},\overline{x}) = \begin{cases} \max\left\{y(\underline{x},\kappa), y(x,\kappa)\right\} & x \leq \overline{x} \\ \max\left\{y(\overline{x},\kappa), (1-\gamma)x - l\right\} & x > \overline{x} \end{cases}, \tag{26}$$

where $\underline{x} = \widehat{x}_-$ and $\overline{x} = y(\varphi_1/(2\varphi_2),\kappa)$. We now show that there exists a non empty set of values of $\kappa$ such that $Y(x;\kappa,\underline{x},\overline{x})$ is an equilibrium investment function with the properties stated in Proposition 1. We start with a technical lemmata. Lemma 1.1 shows that there is a non empty set:

$$\mathcal{C}^* = \left[\frac{4 - (\varphi_1-1)^2}{4\varphi_2}, \frac{9 - (\varphi_1-1)^2}{4\varphi_2}\right],$$

such that for any $\kappa \in \mathcal{C}^*$, $Y(x;\kappa,\underline{x},\overline{x})$ maps $X^* = [\underline{x},\overline{x}]$ into itself:

**Lemma 1.1.** *Let $\kappa \in \mathcal{C}^*$, then for any $x \in [\underline{x},\overline{x}]$, $Y(x;\kappa,\underline{x},\overline{x}) \in [\underline{x},\overline{x}]$.*

**Proof.** Assume $\kappa \in \mathcal{C}^*$, we show here that for any $x \in [\underline{x},\overline{x}]$, $Y(x;\kappa,\underline{x},\overline{x}) \in [\underline{x},\overline{x}]$. We proceed in two steps.

**Step 1.** We first prove that for $\kappa \in \mathcal{C}^*$, then $[y]^2(\frac{\varphi_1}{2\varphi_2},\kappa) \geq \underline{x}$, where $[y]^k(x,\kappa)$ is the $k$th iteration of $y$, $[y]^k(x,\kappa) = y(y^{k-1}(x,\kappa),\kappa)$. A sufficient condition for $[y]^2(\frac{\varphi_1}{2\varphi_2},\kappa) \geq \underline{x}$ is:

$$\frac{\varphi_1 - \sqrt{\varphi_1^2 + 4\varphi_2(\kappa - \underline{x})}}{2\varphi_2} \leq \frac{\varphi_1^2}{4\varphi_2} + \kappa \leq \frac{\varphi_1 + \sqrt{\varphi_1^2 + 4\varphi_2(\kappa - \underline{x})}}{2\varphi_2}. \tag{27}$$

Note that $\kappa \geq \frac{1}{4\varphi_2}\left[4-(\varphi_1-1)^2\right]$ implies $\frac{\varphi_1^2}{4\varphi_2}+k \geq \frac{\varphi_1}{2\varphi_2}$. It follows that the first inequality in (27) is always satisfied. So we need:

$$\frac{\varphi_1^2}{4\varphi_2}+\kappa-\frac{\varphi_1+\sqrt{\varphi_1^2+4\varphi_2(\kappa-\underline{x})}}{2\varphi_2} \leq 0. \tag{28}$$

We now show that this condition is satisfied for any $\kappa \in \left[\frac{4-(\varphi_1-1)^2}{4\varphi_2}, \frac{9-(\varphi_1-1)^2}{4\varphi_2}\right]$. To this goal we proceed by induction:

**Step 1.1.** Given $\kappa \geq \frac{1}{4\varphi_2}\left[4-(\varphi_1-1)^2\right]=\kappa_0$, we have: $\underline{x}=\frac{\varphi_1-1-\sqrt{(\varphi_1-1)^2+4\varphi_2\kappa}}{2\varphi_2} \leq \frac{\varphi_1-3}{2\varphi_2}$. So $[y]^2\left(\frac{\varphi_1}{2\varphi_2},\kappa\right) \geq \underline{x}$, if

$$\frac{\varphi_1^2}{4\varphi_2}+\kappa \leq \frac{\varphi_1+\sqrt{\varphi_1^2+4\varphi_2(\kappa-\frac{\varphi_1-3}{2\varphi_2})}}{2\varphi_2}. \tag{29}$$

After a change in variable, (29) can be written as $\xi^2-2\xi-6 \leq 0$, where $\xi=\sqrt{\varphi_1^2+4\varphi_2(\kappa-\frac{\varphi_1-3}{2\varphi_2})}$. It follows that we need: $\sqrt{\varphi_1^2+4\varphi_2(\kappa-\frac{\varphi_1-3}{2\varphi_2})} \leq 1+\sqrt{7}$, or $\kappa \leq \left[3+2\sqrt{7}-(\varphi_1-1)^2\right]/4\varphi_2$. We therefore conclude that $[y]^2\left(\frac{\varphi_1}{2\varphi_2},\kappa\right) \geq \underline{x}$ is satisfied for any: $\kappa \in \left[\frac{4-(\varphi_1-1)^2}{4\varphi_2}, \frac{3+2\sqrt{7}-(\varphi_1-1)^2}{4\varphi_2}\right]$, which gives us a nonempty set.

**Step 1.2.** We now prove that if $[y]^2\left(\frac{\varphi_1}{2\varphi_2},\kappa\right) \geq \underline{x}$ in for any $\kappa \in \left[\frac{4-(\varphi_1-1)^2}{4\varphi_2},\kappa_n\right]$ for some $\kappa_n < \frac{9-(\varphi_1-1)^2}{4\varphi_2}$, then we can find a $\kappa_{n+1}>\kappa_n$ such that the property $[y]^2\left(\frac{\varphi_1}{2\varphi_2},\kappa\right) \geq \underline{x}$ is satisfied in $\kappa \in \left[\frac{4-(\varphi_1-1)^2}{4\varphi_2},\kappa_{n+1}\right]$. From the previous step, we know that this property is true for $\kappa_1 = \frac{3+2\sqrt{7}-(\varphi_1-1)^2}{4\varphi_2}$. Let us assume we have proven it up to some $\kappa_n \in \left[\frac{3+2\sqrt{7}-(\varphi_1-1)^2}{4\varphi_2}, \frac{9-(\varphi_1-1)^2}{4\varphi_2}\right)$. Note that if $\kappa \geq \kappa_n$, then we have: $\underline{x} \leq \frac{\varphi_1-1-\sqrt{(\varphi_1-1)^2+4\varphi_2\kappa_n}}{2\varphi_2}=\frac{\varphi_1-2-S_n}{2\varphi_2}$, where $S_n=1-\sqrt{(\varphi_1-1)^2+4\varphi_2\kappa_n}$ is the slope of $y(x,\kappa_n)$ at the larger fixed-point $\widehat{x}_+^n$.

Note that (28) is implied by:

$$\varphi_1^2+4\varphi_2\left(\kappa-\frac{\varphi_1-2+S_n}{2\varphi_2}\right)-2\left(2-S_n\right)-2\sqrt{\varphi_1^2+4\varphi_2(\kappa-\frac{\varphi_1-2+S_n}{2\varphi_2})} \leq 0.$$

After a change in variable, this condition can be written as

$$\xi^2 - 2\xi - 2(2 - S_n) \leq 0,$$
$$\Leftrightarrow \quad \xi \leq 1 + \sqrt{1 + 2(2 - S_n)}$$

where $\xi = \sqrt{\varphi_1^2 + 4\varphi_2(\kappa - \frac{\varphi_1 - 2 + S_n}{2\varphi_2})}$. So we need:

$$\sqrt{\varphi_1^2 + 4\varphi_2(\kappa - \frac{\varphi_1 - 2 + S_n}{2\varphi_2})} \leq 1 + \sqrt{1 + 2(2 - S_n)}$$
$$\Leftrightarrow \quad \kappa \leq \frac{3 + 2\sqrt{3 + 2\sqrt{(\varphi_1 - 1)^2 + 4\varphi_2\kappa_n}} - (\varphi_1 - 1)^2}{4\varphi_2} = \kappa_{n+1}.$$

We have the result if $\kappa_{n+1} > \kappa_n$. For this we need: $3 + 2\sqrt{3 + 2\sqrt{(\varphi_1 - 1)^2 + 4\varphi_2\kappa_n}} - (\varphi_1 - 1)^2 > 4\varphi_2\kappa_n$. It is easy to see that this inequality is satisfied for $4\varphi_2\kappa_n + (\varphi_1 - 1)^2 \leq 9$, or: $\kappa_n < \left[9 - (\varphi_1 - 1)^2\right]/(4\varphi_2)$, which is always satisfied since we are assuming it in the induction step.

**Step 1.3.** The sequence $\kappa_n$ is bounded above by $\left(9 - (\varphi_1 - 1)^2\right)/(4\varphi_2)$, thus it converges to $\kappa_\infty = \left(9 - (\varphi_1 - 1)^2\right)/(4\varphi_2)$. Since $[y]^2(\frac{\varphi_1}{2\varphi_2}, \kappa)$ is continuous in $\kappa$, we have that $[y]^2(\frac{\varphi_1}{2\varphi_2}, \kappa) \geq \underline{x}$ for any $\kappa \in \left[\left(4 - (\varphi_1 - 1)^2\right)/(4\varphi_2), \left(9 - (\varphi_1 - 1)^2\right)/(4\varphi_2)\right]$, thus proving the result.

**Step 2.** We now prove that $Y(x;\kappa,\underline{x},\overline{x}) \in X^*$ for any $x$ in $X^* = [\underline{x},\overline{x}]$ and $\kappa$ satisfying $\kappa \in \mathcal{C}^*$. To see this, first note that for any $x \in X^*$, we have $Y(x;\kappa,\underline{x},\overline{x}) \leq \max_z y(z,\kappa) = y(\frac{\varphi_1}{2\varphi_2}, \kappa) = \overline{x}$, where the equality follows from the fact that $y(z,\kappa)$ achieves a maximum at $\varphi_1/(2\varphi_2)$, and the second equality from the definition of $\overline{x}$. Then note that for any $x \in X^*$:

$$Y(x;\kappa,\underline{x},\overline{x}) \geq \min_{z\in\{\underline{x},\overline{x}\}} y(z,\kappa) \geq \min\{y(\overline{x},\kappa), \underline{x}\} \geq \underline{x}, \tag{30}$$

where the first inequality follows from the concavity of $Y(x;\kappa,\underline{x},\overline{x})$ in $X^*$, and the second from $y(\underline{x},\kappa) = \underline{x}$, and the last inequality from $y(\overline{x},\kappa) = [y]^2(\frac{\varphi_1}{2\varphi_2}, \kappa) \geq \underline{x}$ when $\kappa$ satisfies $\kappa \in \mathcal{C}^*$. We conclude that $Y(x;\kappa,\underline{x},\overline{x}) \in X^*$ for any $x$ in $X^* = [\underline{x},\overline{x}]$. ■

It can be verified that when $Y(x;\kappa,\underline{x},\overline{x})$ is the investment function with $\kappa \in \mathcal{C}^*$, then the state will eventually enter in $[\underline{x},\overline{x}]$ in a finite number of steps starting from any point $x \notin [\underline{x},\overline{x}]$. This observation together with Lemma 1.1. implies that when the players strategy is (26), then the state will be in $X^* = [\underline{x}, \overline{x}]$ in all periods except at most for a finite transition period. The next lemma shows that the investment function (26) is actually feasible for all $x$ when $R > R^*(\alpha)$ as defined in (11).

**Lemma 1.2.** *If $R \geq R^*(\alpha)$ as defined in (11) is satisfied, then $Y(x; \kappa, \underline{x}, \overline{x})$ is feasible for all $x$.*

**Proof.** Assume $R \geq R^*(\alpha)$ as defined in (11) in Section 3.1 of the paper, we show here that then $Y(x; \kappa, \underline{x}, \overline{x})$ is feasible for all $x$. Define $x_l^-$, $x_l^+$ the points at which $y$ intersects $(1-\gamma)x - l$, that is:

$$\begin{aligned} x_l^- &= \frac{\varphi_1 - (1-\gamma) - \sqrt{(\varphi_1 - (1-\gamma))^2 + 4\varphi_2 (\kappa + l)}}{2\varphi_2}, \\ x_l^+ &= \frac{\varphi_1 - (1-\gamma) + \sqrt{(\varphi_1 - (1-\gamma))^2 + 4\varphi_2 (\kappa + l)}}{2\varphi_2}. \end{aligned}$$

We have $y(x, \kappa) \geq (1-\gamma)x - l$ for $x \in \left[x_l^-, x_l^+\right]$. Consider $x_l^-$ first. We have:

$$x_l^- - \underline{x} = \frac{1}{2\varphi_2}\left[\gamma + \sqrt{(\varphi_1 - 1)^2 + 4\varphi_2\kappa} - \sqrt{(\varphi_1 - (1-\gamma))^2 + 4\varphi_2 (\kappa + l)}\right]. \tag{31}$$

We need to have $x_l^- - \underline{x} \leq 0$. Note first that:

$$(\varphi_1 - 1)^2 + 4\varphi_2\kappa \leq (\varphi_1 - (1-\gamma))^2 + 4\varphi_2 (\kappa + l) \Leftrightarrow \gamma^2 + 2\gamma(\varphi_1 - 1) + 4\varphi_2 l \geq 0. \tag{32}$$

Assume first that (32) is satisfied. In this case, the square parenthesis in (31) is increasing in $\kappa$ and $x_l^- - \underline{x}$ can be bounded above inserting the upperbound of $\mathcal{C}^*$:

$$\begin{aligned} x_l^- - \underline{x} &\leq \frac{1}{2\varphi_2}\left[\gamma + \sqrt{(\varphi_1 - 1)^2 + 9 - (\varphi_1 - 1)^2} - \sqrt{(\varphi_1 - (1-\gamma))^2 + 4\varphi_2 l + 9 - (\varphi_1 - 1)^2}\right] \\ &= \frac{1}{2\varphi_2}\left[\gamma + 3 - \sqrt{9 + \gamma^2 + 2\gamma(\varphi_1 - 1) + 4\varphi_2 l}\right]. \end{aligned}$$

So we have $x_l^- - \underline{x} \leq 0$ if $\varphi_1 \geq 4 - 2\varphi_2 l/\gamma$. Consider now the case: $\gamma^2 + 2\gamma(\varphi_1 - 1) + 4\varphi_2 l < 0$.

Now $x_l^- - \underline{x}$ can be bounded above inserting the lowerbound of $\mathcal{C}^*$:

$$\begin{aligned} x_l^- - \underline{x} &\leq \frac{1}{2\varphi_2}\left[\gamma + \sqrt{(\varphi_1-1)^2 + 4 - (\varphi_1-1)^2} - \sqrt{(\varphi_1-(1-\gamma))^2 + 4\varphi_2 l + 4 - (\varphi_1-1)^2}\right] \\ &= \frac{1}{2\varphi_2}\left[\gamma + 2 - \sqrt{4+\gamma^2+2\gamma(\varphi_1-1)+4\varphi_2 l}\right]. \end{aligned}$$

So we have $x_l^- - \underline{x} \leq 0$ if $\varphi_1 \geq 3 - 2\varphi_2 l/\gamma$. It follows that a sufficient condition is that $\varphi_1 \geq 4 - 2\varphi_2 l/\gamma = \varphi_{11}^*$.

Consider now $x_l^+$. We have:

$$x_l^+ - \overline{x} = \frac{1}{4\varphi_2}\left[2\varphi_1 - 2(1-\gamma) + 2\sqrt{(\varphi_1-(1-\gamma))^2 + 4\varphi_2(\kappa+l)} - \left(\varphi_1^2 + 4\varphi_2\kappa\right)\right].$$

We need $x_l^+(\varphi_1) - \overline{x} \geq 0$. The right hand side is concave in $\kappa$, so it is minimized at one of the extremes. If the minimum is at the lowerbound, we have:

$$\begin{aligned} x_l^+ - \overline{x} &= \frac{1}{4\varphi_2}\left[2\varphi_1 - 2(1-\gamma) + 2\sqrt{(\varphi_1-(1-\gamma))^2 + 4\varphi_2(\kappa+l)} - \left(\varphi_1^2 + 4\varphi_2\kappa\right)\right] \\ &\geq \frac{1}{4\varphi_2}\left[-5 + 2\gamma + 2\sqrt{\gamma^2 + 2(\varphi_1-1)\gamma + 4 + 4\varphi_2 l}\right]. \end{aligned}$$

It follows that $x_l^+ - \overline{x} \geq 0$ if $\varphi_1 \geq \frac{3}{8\gamma}[3-4\gamma] - 2\frac{\varphi_2 l}{\gamma}$. If the minimum is at the upperbound, we have:

$$x_l^+ - \overline{x} \geq \frac{1}{4\varphi_2}\left[-10 + 2\gamma + 2\sqrt{\gamma^2 + 2(\varphi_1-1)\gamma + 9 + 4\varphi_2 l}\right].$$

Which can be written as $\varphi_1 \geq \frac{1}{\gamma}[8-4\gamma] - 2\frac{\varphi_2 l}{\gamma}$. It follows that a sufficient condition for $x_l^+ - \overline{x} \geq 0$ is that $\varphi_1 \geq \frac{1}{\gamma}[8-4\gamma] - 2\frac{\varphi_2 l}{\gamma} = \varphi_{12}^*$. Note that $\varphi_{12}^* - \varphi_{11}^* = 8(1/\gamma - 1) > 0$. We conclude that $Y(x;\kappa,\underline{x},\overline{x})$ is feasible for all $x \in [\underline{x},\overline{x}]$ if $\varphi_1 \geq \varphi_{12}^*$. Using the definitions of $\varphi_1$ and $\varphi_2$, the condition becomes:

$$\begin{aligned} \frac{1}{1-\alpha}\left[\frac{2}{\delta} - (1+\alpha)(1-\gamma) + \frac{2\beta}{K}\widehat{x}\right] &\geq \frac{1}{\gamma}[8-4\gamma] - 2\frac{l}{\gamma}\cdot\frac{\beta}{(1-\alpha)K} \\ \Leftrightarrow\ \frac{\beta}{K} &\geq \frac{4\delta(1-\alpha)(2-\gamma) + \delta(1+\alpha)(1-\gamma)\gamma - 2\gamma}{2\delta(\widehat{x}\gamma + l)} = R^*(\alpha). \end{aligned}$$

It follows that $y(x,\kappa) \geq (1-\gamma)x - l$ for $x \in [\underline{x},\overline{x}]$ if $R \geq R^*(\alpha)$. Moreover $Y(x;\kappa,\underline{x},\overline{x})$ obviously satisfies the constraint $y \geq (1-\gamma)x - l$ for $x > \overline{x}$. Finally we have that $(1-\gamma)x - l \leq (1-\gamma)\underline{x} - l \leq y(\underline{x},\kappa) = Y(x;\kappa,\underline{x},\overline{x})$ in $x < \underline{x}$. ■

We now show that $Y(x;\kappa,\underline{x},\overline{x})$ is an optimal policy for an incumbent. The objective function of (5) can be written as $P(x_{t+1}) = Kx_{t+1} + \delta v(x_{t+1})$, since $y = x_{t+1}$ and $x_t = x$ is given for the incumbent at $t$. In the following, we write the objective function as a generic state $x$ as $P(x) = Kx + \delta v(x)$. We now show that given $Y(x;\kappa,\underline{x},\overline{x})$, the objective function $P(x)$ of (5) is concave; almost everywhere differentiable; and differentiable in $X^*$, implying that the first order necessary condition with respect to $y = x_{t+1}$ is sufficient for optimality in problem (5).[31] To this goal, first note that in $X^*$, we have $Y(x;\kappa,\underline{x},\overline{x}) = y(x,\kappa)$, which is differentiable. This implies that $P(x)$ is differentiable in this interval. The objective function $P(x)$ is also obviously differentiable in $x < \underline{x}$ and $x > \overline{x}$. In $[\underline{x},\overline{x}]$ the derivative objective function is such that:

$$P'(x) = K + \delta v'(x) = K - \delta\left[e'(x) + \frac{(1+\alpha)K(1-\gamma)}{2} + \frac{(1-\alpha)K}{2}y'(x,\kappa)\right] = 0.$$

Consider now for $x > \overline{x}$. Note that:

$$\overline{x} = y(\frac{\varphi_1}{2\varphi_2},\kappa) \geq \frac{1}{4\varphi_2}\left[\varphi_1^2 + 1 - (\varphi_1 - 1)^2\right] = \frac{\varphi_1}{2\varphi_2} = x^*$$

where $x^* = \arg\max_z y(z,\kappa)$. Since $\overline{x} \geq \varphi_1/(2\varphi_2)$, $y(x,\kappa)$ is concave, and $y'(\varphi_1/(2\varphi_2),\kappa) = 0$, we conclude that $y'(\overline{x},\kappa) \leq 0$. We therefore have:

$$\begin{aligned} P'(x) &\leq K - \delta\left[e'(x) + \frac{(1+\alpha)K(1-\gamma)}{2}\right] \\ &\leq K - \delta\left[e'(x) + \frac{(1+\alpha)K(1-\gamma)}{2} + \frac{(1-\alpha)K}{2}y'(x,\kappa)\right] = 0 \end{aligned} \tag{33}$$

for any $x > \overline{x}$. Naturally, $\underline{x} < \varphi_1/(2\varphi_2)$, so for $x < \overline{x}$:

$$P'(x) = K - \delta\left[e'(x) + \frac{(1+\alpha)K(1-\gamma)}{2}\right] \geq K - \delta\left[\begin{array}{c} e'(x) + \frac{(1+\alpha)K(1-\gamma)}{2} \\ +\frac{(1-\alpha)K}{2}y'(x,\kappa) \end{array}\right] = 0. \tag{34}$$

Conditions (33) and (34) imply that $P(x)$ achieves a maximum at any point in $X^*$. To see the concavity of $P(x)$, note that it is continuous, concave with positive derivative in $x < \underline{x}$, flat in

[31] Note that for problem (5), the choice variable is $y = x_{t+1}$. We write the objective function of this problem as $P(x)$.

$x \in [\underline{x}, \overline{x}]$, and concave with negative derivative in $x > \overline{x}$. To see that $Y(x; \kappa, \underline{x}, \overline{x})$ is an optimal policy for the incumbent note that (33) and (34), $P(x)$ achieves a maximum in $X^* = [\underline{x}, \overline{x}]$ and that, when $\kappa \in \mathcal{C}^*$, $Y(x; \kappa, \underline{x}, \overline{x}) \in X^*$ for any $x$ for which it is feasible; and given the state $x$ a point $y$ in $X^*$ is not feasible, the policy is at a constrained optimum.

We finally show that for any $\kappa \in \mathcal{C}^*$, $Y(x; \kappa, \underline{x}, \overline{x})$ does not admit a stable steady state. Note that $Y(x; \kappa, \underline{x}, \overline{x}) = x$ at the point: $\widehat{x}_+ = \left[(\varphi_1 - 1) + \sqrt{(\varphi_1 - 1)^2 + 4\varphi_2\kappa}\right]/2\varphi_2$. We have $Y'(\widehat{x}_+; \kappa, \underline{x}, \overline{x}) = 1 - \sqrt{(\varphi_1 - 1)^2 + 4\varphi_2\kappa} \leq 0$, so $\widehat{x}_+$ is an unstable steady state. ■

## 7.2 Proof of Lemma 1

We proceed in three steps.

**Step 1.** We first observe that $y_c = y(x, c)$, as defined in (10), is conjugate to $Q_k = x^2 + k$ for $k = (\varphi_1/2)(1 - \varphi_1/2) - \varphi_2 c$ by the homeomorphism $\xi(x) = \varphi_1/2 - \varphi_2 x$. To see this note that $\xi \circ y_c(x) = \varphi_1/2 + \varphi_2^2 x^2 - \varphi_1\varphi_2 x - c\varphi_2$, moreover:

$$\begin{aligned} Q_k \circ \xi(x) &= [\varphi_1/2 - \varphi_2 x]^2 + (\varphi_1/2)(1 - \varphi_1/2) - \varphi_2 c \\ &= \varphi_1/2 + \varphi_2^2 x^2 - \varphi_2(\varphi_1 x + c) = \xi \circ y_c(x). \end{aligned}$$

So we have $Q_k \circ \xi = \xi \circ y_c$. Similarly, we can show that $L_\eta$ is conjugate to $Q_k$ with $k = \eta/2(1 - \eta/2)$ by the homeomorphism $h_\eta = -\eta x + \eta/2$.

**Step 2.** Let us now define $c(\eta; \varphi_1, \varphi_2)$ such that $\eta/2(1 - \eta/2) = (\varphi_1/2)(1 - \varphi_1/2) - \varphi_2 \cdot c(\eta; \varphi_1, \varphi_2)$, that is:

$$c(\eta; \varphi_1, \varphi_2) = \frac{1}{\varphi_2}\left[(\varphi_1/2)(1 - \varphi_1/2) - \eta/2(1 - \eta/2)\right].$$

We can then write:

$$\begin{aligned} L_\eta &= h_\eta^{-1} \circ Q_{\eta/2(1-\eta/2)} \circ h_\eta = h_\eta^{-1} \circ \left[\xi \circ y_{c(\eta;\varphi_1,\varphi_2)} \circ \xi^{-1}\right] \circ h_\eta \\ &= \left[h_\eta^{-1} \circ \xi\right] \circ y_{c(\eta;\varphi_1,\varphi_2)} \circ \left[\xi^{-1} \circ h_\eta\right] = \left[\xi^{-1} \circ h_\eta\right]^{-1} \circ y_{c(\eta;\varphi_1,\varphi_2)} \circ \left[\xi^{-1} \circ h_\eta\right] \\ &= z_\eta^{-1} \circ y_{c(\eta;\varphi_1,\varphi_2)} \circ z_\eta \Leftrightarrow z_\eta \circ L_\eta = y_{c(\eta;\varphi_1,\varphi_2)} \circ z_\eta \end{aligned}$$

where $z_\eta = \xi^{-1} \circ h_\eta$. This implies that $L_\eta$ is topologically conjugate to $y(x, c(\eta; \varphi_1, \varphi_2))$ through the homeomorphism $z_\eta$.

**Step 3.** From Proposition 1, $y(x,c)$ with $c = c(\eta; \varphi_1, \varphi_2)$ is an equilibrium if $c(\eta; \varphi_1, \varphi_2) \in \mathcal{C}^*$ as defined in (12). We have:

$$\frac{1}{\varphi_2}\left[(\varphi_1/2)(1-\varphi_1/2) - \eta/2(1-\eta/2)\right] \geq \frac{4-(\varphi_1-1)^2}{4\varphi_2} \Leftrightarrow \eta/2(1-\eta/2) \leq -\frac{3}{4}.$$

Moreover, we need:

$$\frac{1}{\varphi_2}\left[(\varphi_1/2)(1-\varphi_1/2) - \eta/2(1-\eta/2)\right] \leq \frac{9-(\varphi_1-1)^2}{4\varphi_2} \Leftrightarrow \eta/2(1-\eta/2) \geq -2.$$

We can therefore construct an equilibrium that is conjugate to $L_\eta$ if: $-2 \leq \eta/2(1-\eta/2) \leq -\frac{3}{4}$. We conclude that we can construct an equilibrium that is conjugate to $L_\eta$ if $3 \leq \eta \leq 4$ as desired.

■

## 7.3 Proof of Proposition 2

The result follows from the argument in the text. For the existence of values $\eta_k$ in $[3,4]$ such that $L_{\eta_k}$ has a stable cycle of period $k$ or displays topological chaos, see for example Rasband [1990], ch. 2.3. For specific values that generate stable cycles up to the order 11 see Metropolis et al. [1973]. ■

## 7.4 Proof of Proposition 3

The fact that we have a set of positive measures of values in $\mathcal{C}^*$ such that an equilibrium with ergodic distribution exists follows from Lemma 1 and the discussion in Section 3. We proceed to the characterization of the ergodic distribution for $c = c(4; \varphi_1, \varphi_2)$. Let $\mu$ be the measure that is invariant under $L_4$, so that $\mu = L_{4*}\mu$. It is well known that $\mu$ is the arc sine law $\mu(x) = 1/\left(\pi\sqrt{x(1-x)}\right)$ (See, for example, Hilborn [2000, ch. 9.5]). Let us define the so called "push forward" measure $z_{4*}\mu$ by: $z_{4*}\mu(X) := \mu\left(z_4^{-1}(X)\right)$, where $z_4$ is the homeomorphism such that

$y \circ z_4 = z_4 \circ L_4$, defined in the proof of Lemma 1. We have:

$$z_{4*}\mu = z_{4*}\left[L_{4*}\mu\right] = \left(z_4 \circ L_4\right)_* \mu = \left(y \circ z_4\right)_* \mu = y_*\left(z_{4*}\mu\right)$$

where in the second and fourth equalities we use the definition of the push forward measure, and in the third the fact that $y \circ z_4 = z_4 \circ L_4$. So we have: $y_*\left(z_{4*}\mu\right) = z_{4*}\mu$. To find $z_{4*}\mu$ note that $z_4 = \left(\varphi_1 - 4\right)/\left(2\varphi_2\right) + \left(4/\varphi_2\right)x$, so $x = \varphi_2 z_4/4 - (\varphi_1 - 4)/8$. Using the Perron-Frobenius operator, it follows that:

$$\begin{aligned} \mu^*(x,\alpha,R) &= \frac{1}{\pi\sqrt{x(1-x)}\,|z_4'(x)|} = \frac{b(\alpha,R)}{4\pi\cdot\sqrt{\left(\frac{\varphi_2}{4}x - \frac{\varphi_1-4}{8}\right)\left(1-\frac{\varphi_2}{4}x+\frac{\varphi_1-4}{8}\right)}} \\ &= \frac{2R}{\pi\left(1-\alpha\right)\cdot\sqrt{16-\left(\frac{2R}{(1-\alpha)}x - \frac{1}{1-\alpha}\left[\frac{2}{\delta}-\left(1+\alpha\right)\left(1-\gamma\right)+2R\widehat{x}\right]\right)^2}}. \end{aligned}$$

Which gives us (14) in the statement of Proposition 3. ■

# 8 Appendix B: Results presented in Section 4

In Section 3.1 we present the sufficient condition for the existence of chaotic equilibria when the cost function is a generic convex function $e(x)$, discussed in Section 4 of the paper. In Section 3.2 we describe the algorithm used to solve (15) in Section 4 of the paper and compute the equilibria presented in Figure 3 of the paper.

## 8.1 A sufficient condition for chaos with $e(x)$

The necessary condition (9) in the main text, implies:

$$y(x,c) = \left[2/\delta - \left(1+\alpha\right)\left(1-\gamma\right)\right]/\left(1-\alpha\right)\cdot x - (2/K)e(x) + c, \tag{35}$$

where $c$ is the constant of integration that defines the possible solutions. When $e(x)$ is quadratic, this function coincide with (10) in the main text. In this section we present a general condition under which $y(x,c)$ displays chaotic behavior when $e(x)$ is a general cost function.

As in Section 3, assume that $e(x)$ is a differentiable, with continuous derivative, and strictly concave in $x$. Under these assumptions, $y(x,c)$ is differentiable and $C^1$-unimodal in $[\underline{x}, \overline{x}]$ for any $c$ such that $\overline{x} \geq x^*$ (where we define $x^*$, $\overline{x} = f(x^*, c)$ and $\underline{x} = x^*_-(c)$ to be, respectively, the maximizer, the maximum and the smaller fixed point of $y(x,c)$).[32] The sufficient condition presented in Proposition A1 below requires a relatively mild strengthening of the assumptions on $e(x)$. A dynamical system $y$ is said to be *S-unimodal* if it is unimodal and it has nonpositive Schwarzian derivative at all noncritical points (i.e. except for $x^*$ such that $f'(x^*) = 0$).[33] The solution $y(x,c)$ in (35) is S-unimodal if $e(x)$ is S-unimodal. Many common functions satisfy this condition including, for instance, any polynomial of degree larger than or equal to 2 with real valued critical points (and thus the quadratic used in Section 4) and the exponential function. The following result provides a simple sufficient condition guaranteeing that the solution of (35) displays ergodic chaos in $[\underline{x}, \overline{x}]$ when $e(x)$ is S-unimodal.[34] Define $\Delta^{2,3}(c) = [y]^3(x^*, c) - [y]^2(x^*, c)$, this is the gap between the third and the second iteration starting from the modal point $x^*$ (i.e., the point at which the maximum is attained), it is an easily computed function of $c$ given $y(x,c)$. We have:

**Proposition A1.** *Assume that $y(x,c)$ in (35) has negative Schwarzian derivative with respect to $x$, and that there is a $c'$ and $c''$ with $c'' > c'$ such that $\Delta^{2,3}(c') \cdot \Delta^{2,3}(c'') \leq 0$ and $[y^*]'([y^*]^2(x^*, c''), c'') > 1$, then there exists a $c^*$ such that $y(x^*, c^*)$ displays ergodic chaos on $[\underline{x}, \overline{x}]$.*

**Proof.** The function $y(x;c)$ defined in (35) is strictly concave and $C^1$-unimodal in $[x_-(c), y(x^*, c)]$

[32] A strictly concave, $C^1$ function is $C^1$ unimodal in a set $[a, b]$ if $f(x^*) \geq x^*$ at its critical point $x^*$. Note that the function $y(x;c)$ in (35) has a unique maximizer at $x^*$ independent of $c$, and two fixed-points $x^*_-(c)$, $x^*_-(c)$ with $x^*_-(c) < x^* < x^*_-(c)$.

[33] For a definition of S-unimodal functions and the Schwarzian derivative, see Collet and Eckmann [1980].

[34] Li and Yorke [1975] have shown that, if continuous in $x$, a dynamical system $y(x)$ has cycles of any order in a set $X$ if there is a $x' \in X$ such that: $[y]^3(x') < x' < y(x') < [y]^2(x')$. Li and Yorke's condition is relatively easy to verify (examples are presented in the working paper Battaglini [2021]), but it does generally guarantee that the regions in which $y(x;c^*)$ is chaotic (the scrambling set) has positive measure, leaving open the possibility that the chaotic region is reached from no initial condition except a measure zero states.

for any $c$. Moreover by the assumptions of the proposition, it is trice continuously differentiable with negative Schwartzian derivative in $[x_-(c), y(x^*, c)]$ for any $c$. This implies that, for any $c$, it satisfies assumptions S1-S3 and S5 of Corollary 6 in Grandmont [1992]. By continuity and the fact that $\Delta^{2,3}(c') \cdot \Delta^{2,3}(c'') \leq 0$, there must be a $c^* \in [c', c'']$ such that $[y^*]^3(x^*, c^*) = [y^*]^2(x^*, c^*)$, thus implying that the second iteration starting from the critical point $x^*$ is a steady state of $y(x; c^*)$. From strict concavity and $[y^*]'([y^*]^2(x^*, c''), c'') > 1$, we have:

$$[y^*]'([y^*]^2(x^*, c^*), c^*) > [y^*]'([y^*]^2(x^*, c''), c'') > 1.$$

So $y(x; c^*)$ enters an unstable cycle of period 1 at the second iteration starting from $x^*$. Note moreover that $y(x; c^*) > x$ for all $x \in [x_-(c), x^*]$, thus it satisfies assumption $S4''$ of Corollary 6 in Grandmont [1992]. By this result, we therefore conclude that $y(x; c^*)$ has a unique absolutely continuous invariant ergodic measure in $[x_-(c), y(x^*, c)]$. ■

The following examples show that the condition of Proposition A1 is easy to apply in specific examples. Our first example derives the existence of an equilibrium with ergodic chaos using Proposition A1 for the economy of the examples in Figure 1 and 2 in the main text.[35]

**Example 3.1 (Quadratic).** As in the examples of Figure 1 and 2, assume $e(x) = (\beta/2)(x - x)^2$ and $\alpha = .8$, $\delta = .95$, $\gamma = .5$, $l = .5$. It is easy to verify that for $c'' = 11.78$, $\Delta^{2,3}(c'')$ is $0.164\,78 > 0$; and for $c' = 11.6$ it is equal to $\Delta^{2,3}(c') = -5.4138 < 0$. Moreover, $[y^*]'([y^*]^2(x^*, c''), c'') = 1.8785$, so the conditions of Proposition A1 are verified.[36]

The next example allows for a completely different functional form, the exponential.

**Example 3.2 (Exponential).** Assume, as in Example 3.1, that $\alpha = .8$, $\delta = .95$, $\gamma = .5$, $l = .5$,

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[35] Propositions 2-3 are less general than Proposition A1 as they require the more demanding assumption that $e(x) = (\beta/2)(x - x)^2$. These proof, however, allows us to reach a much more precise characterization of the possible dynamics, as discussed in Section 4.

[36] Indeed in correspondence to $c^* = 11.739$, we have the equilibrium of Figure 1 and 2 that is topologically conjugate to the Ulam and von Neumann [1947]'s example. This equilibrium corresponds to the point $c^* = 11.739$ in correspondence of which we solve $\Delta^{2,3}(c^*) = 0$.

but $e(x) = -2e^x$. Again, using (35) it is easy to verify that for $c' = 5.58$, $\Delta^{2,3}(c') = 0.50003 > 0$; and for $c'' = 5.8$, $\Delta^{2,3}(c'') = -.282\,3 < 0$. Moreover, $[y^*]'\left([y^*]^2(x^*, c''), c''\right) = 4.3056 > 1$. The equilibrium is found in closed form setting $c^*$ such that $\left[[y^*]^3(x^*, c^*) - [y^*]^2(x^*, c^*)\right] = 0$, so $c^* = 5.6924515$, implying $y(x, c) = 6.026\,3x - 5.0e^x + 5.6924515$.

Proposition A1 extends a theorem by Misiurewicz [1981] using the equilibrium construction leading to (35). Misiurewicz [1981] proved that if a dynamical system $y$ is $S$-unimodal and such that the iterates $[y]^n(x^*)$ of the critical point $x^*$ converge to an unstable cycle, then $y$ has exactly one absolutely continuous invariant measure.[37] Clearly this is a knife edge condition that defines a nongeneric set of functions. In our case we can construct equilibria with ergodic chaos that exist for generic parametrizations because $c$ is endogenous and it can generically be selected to make sure that there is an $n$ and a $c^*$ such that $[y]^n(x^*, c^*)$ enters an unstable cycle. We should moreover note that the condition in Proposition A1 is only sufficient and indeed it can be easily extended using the same logic if we are willing to check conditions on higher iterates of $y(x; c)$.[38]

## 8.2 Construction of the equilibria in Figure 3

### 8.2.1 The example in Panels A.1 and A.2

In Figure 3 we assume the preferences described in (21) in Section 4 under the assumption that the elasticity parameter is $\xi = 5\%$ and the following parametrization of the economy:

$$\begin{array}{lllll} \alpha = .7 & \delta = .85 & \gamma = .07 & \beta = .75 & \theta_1 = 2 \\ \theta_2 = 1 & K = .4 & \widehat{x} = .15 & l = .5 & X = [0, 0.55] \end{array} \tag{36}$$

To compute the equilibrium, consider first the same parametrization, but with $\xi = 0$. In this case, Proposition 1 characterize the equilibria in closed form. The solution of (9) in Section 3.1

[37] Misiurewicz [1981] actually proved that if $y$ has no weakly stable orbit, and there exists an open neighborhood of $U$ of $x^*$ such that $[v]^n(x) \notin U$, then $y$ has exactly one absolutely continuous invariant measure. The statement above is a corollary of this result (see Corollary 6 in Grandmont [1992] and Theorem II.8.3 and Corollary II.8.4 in Collet and Eckmann [1980]).

[38] The condition in Proposition A1 is indeed sufficient to have the second iterate enter an unstable cycle. It is easy to verify that the second example in Figure 1 (the one constructed with Ruelle's constant) has the property that it is the third iterate to enters an unstable cycle.

of the paper is:

$$y(x, \kappa) = \varphi_1 x - \varphi_2 x^2 + \kappa$$

where $\varphi_1$ and $\varphi_2$ are defined in Section 3.1 of the paper as function of the fundamentals. If we set $\kappa$ equal to $c(\mu_2, a, b)$ where $\mu_2 = 3.1$, then $y(x, c(\mu_2, a, b))$ defines a cycles of period 2, since it is topologically conjugated to $L_{\mu_2}(x)$. The periodic points of this cycle are $x_1 = 0.1290$, $x_2 = 0.5066$, both solutions of the equation $[y]^2(x, c(\mu_2, a, b)) = x$. We can moreover verify that there are bounds $\underline{x}$, $\overline{x}$ such that $y(x, \kappa) \in (\underline{x}, \overline{x})$ for all $x \in X$.

To construct the equilibrium in Panels A.1 and A.2 of Figure 3, we proceed as follows. In Step 1, we solve numerically for a solution $y^{\xi}(z, x)$ to the PDE (15) in Section 4 of the paper. We then define a strategy as:

$$Y^{\xi}(z, x) = \begin{cases} \max\left\{y^{\xi}(\overline{x}; x), (1-\gamma)z - l\right\} & z > \overline{x} \\ y^{\xi}(z; x) & z \in [\underline{x}, \overline{x}] \\ y^{\xi}(\underline{x}; x) & z < \underline{x} \end{cases} \tag{37}$$

for all $x \in X$. Since preferences in (21) in the paper converge to the quasi-linear preferences, $y^{\xi}(z, x)$ converge to $y(x, \kappa_2)$ as $\xi \to 0$ and thus $y^{\xi}(z, x)$ and $Y^{\xi}(z, x)$ are self maps in $[\underline{x}, \overline{x}]$ (i.e. $y^{\xi}(\cdot)$ and $Y^{\xi}(\cdot)$ map $[\underline{x}, \overline{x}]$ to itself) for a sufficiently small $\xi$. We verify this property numerically for parametrization (36) when $\xi = 5\%$ for $z \in [\underline{x}, \overline{x}]$ and $x \in X$. In Step 2 we prove that if we define a strategy $Y^{\xi}(z, x)$ as in (37), then the objective function is constant and maximal in the set $[\underline{x}, \overline{x}]$, which implies that $Y^{\xi}(x_t, x_{t-1})$ is indeed an optimal response for a policy maker in state $(x_t, x_{t-1})$.

**Step 1: Solving the PDE.** Assume preferences as in Example 7.1. The PDE (15) in Section 4 of the paper becomes:

$$\begin{aligned} &\frac{K(z-(1-\gamma)x)^{-\xi}}{\delta} - (1-\gamma)K(H+y(z,x)-(1-\gamma)z)^{-\xi} - \beta\,(z-\widehat{x}) \\ &\quad -\frac{1}{2}\,(1-\alpha)\,K(H+y(z,x)-(1-\gamma)z)^{-\xi}\cdot y_1(z,x) \\ &\quad +\frac{1}{2}\,(1-\alpha)\,K(1-\gamma)(H+y(z,x)-(1-\gamma)z)^{-\xi} \\ &-\frac{1}{2}\,(1-\alpha)\,\delta K \begin{pmatrix} H+y(y(z,x),z) \\ -(1-\gamma)y(z,x) \end{pmatrix}^{-\xi} \cdot y_2(y(z,x),z) = 0 \end{aligned} \tag{38}$$

When $\xi = 0$, this PDE collapses to a standard differential equation with a solution $y(x,\kappa)$ that is quadratic in $z$, independent of $x$ and parametrized by a constant of integration $\kappa$. For small values of $\xi$, the solution can be approximated as a polynomial $\widehat{y}(z,x;\boldsymbol{\omega})$ or order $m$ in $z,x$, with parameters described by a vector $\boldsymbol{\omega}$. Let $\Delta\,(z,x;\boldsymbol{\omega})$ be the right hand side of (38) evaluated at $\widehat{y}(z,x;\boldsymbol{\omega})$. The solution is computed by selecting $\omega$ to minimize $\sum_{z,x\in X} \|\Delta\,(z,x;\boldsymbol{\omega})\|$ under the constraint that $\widehat{y}(x_1,x_2;\boldsymbol{\omega}) = x_2$ and $\widehat{y}(x_2,x_1;\boldsymbol{\omega}) = x_1$. We find a solution with precision in the order of $10^{-25}$ with a polynomial of second order degree. The fact that $y^{\xi}(z,x) = \widehat{y}(z,x;\boldsymbol{\omega})$ and $Y^{\xi}(z,x)$ are a self maps in $[\underline{x},\overline{x}]\times X$ is verified numerically computing the maximum and minimum of the function in $[\underline{x},\overline{x}]\times X$. Panels A.1 and A.2 in Figure 3 illustrate the function $y^{\xi}(z,x)$ in the $z,x$ space (the shaded surface).

**Step 2**. **Optimality of the best responses.** Let $Y^{\xi}(z,x)$ be defined by (37), where $y^{\xi}(z,x)$ solves (38). We now prove that if a policy maker in state $x$ expects future policy makers to follow $Y^{\xi}(z,x)$ if $z$ is chosen, then s/he finds it optimal to follow a strategy $z = Y^{\xi}(x,x')$ for any possible state $x' \in X$ preceding $x$. A policy maker's objective function can be written as: $P_{\xi}(z,x) = u(z-(1-\gamma)x,x) + \delta v(z,x)$. We now show that this function is almost everywhere differentiable, achieving its maximum at any point $z \in [\underline{x},\overline{x}]$. Define:

$$\begin{aligned} \left[Y^{\xi}\right]^2(z,x) &= Y^{\xi}(Y^{\xi}(z,x),z) \\ \left[Y^{\xi}\right]^k(z,x) &= Y^{\xi}(\left[Y^{\xi}\right]^{k-1}(z,x),z) \end{aligned}$$

Similarly we can define the $k$th iterate of $y^{\xi}(z;x)$, $\left[y^{\xi}\right]^{k}(z;x) = y^{\xi}(\left[y^{\xi}\right]^{k-1}(z;x);z)$. First note that we have $\left[Y^{\xi}\right]^{k}(z,x) = \left[y^{\xi}\right]^{k}(z;x)$ for all $z, x \in [\underline{x}, \overline{x}] \times X$, so $P_{\xi}(z,x)$ is differentiable in $z$ for $z \in (\underline{x}, \overline{x})$. We moreover can write:

$$P'_{\xi}(z,x) = u_1(z-(1-\gamma)x,x) + \delta \begin{bmatrix} u_2(y^{\xi}(z;x),z) - \Phi_2\left(y^{\xi}(z;x),z\right) \\ -\Phi_1\left(y^{\xi}(z;x),z\right) y_1^{\xi}(z;x) \\ -\delta\Phi_1\left(y^{\xi}(y^{\xi}(z;x);z), y^{\xi}(z;x)\right) y_2^{\xi}(y^{\xi}(z;x);z) \end{bmatrix} = 0$$

for all $z \in (\underline{x}, \overline{x})$ by the definition of $y^{\xi}(z,x)$.

Consider now $z < \underline{x}$. Since it must be that $\underline{x} < \varphi_1/(2\varphi_2)$, there is a $\xi_1$ such that $y_1^{\xi}(z;x) \geq 0$ for all $z < \underline{x}$ and $x \in X$ for $\xi \leq \xi_1$. Therefore, we can write:

$$\begin{aligned} P'_{\xi}(z,x) &= u_1(z-(1-\gamma)x,x) + \delta \begin{bmatrix} u_2(y^{\xi}(z;x),z) - \Phi_2\left(y^{\xi}(z;x),z\right) \\ -\delta\Phi_1\left(y^{\xi}(y^{\xi}(z;x);z), y^{\xi}(z;x)\right) y_2^{\xi}(y^{\xi}(z;x);z) \end{bmatrix} \\ &\geq u_1(z-(1-\gamma)x,x) + \delta \begin{bmatrix} u_2(y^{\xi}(z;x),z) - \Phi_2\left(y^{\xi}(z;x),z\right) \\ -\Phi_1\left(y^{\xi}(z;x),z\right) y_1^{\xi}(z;x) \\ -\delta\Phi_1\left(y^{\xi}(y^{\xi}(z;x);z), y^{\xi}(z;x)\right) y_2^{\xi}(y^{\xi}(z;x);z) \end{bmatrix} = 0 \end{aligned}$$

where the inequality follows from the fact that $\Phi_1\left(y^{\xi}(z;x),z\right) \geq 0$ and $y_1^{\xi}(z;x) \geq 0$, and the final equality from the definition of $y^{\xi}(z;x)$. Similarly for $z > \overline{x}$, we have an $\xi_2 \leq \xi_1$ such that for $\xi \leq \xi_2$, $y_1^{\xi}(z;x) \leq 0$ for all $z > \overline{x}$ and $x \in X$, implying that at all points of differentiability:

$$\begin{aligned} P'_{\xi}(z,x) &= u_1(z-(1-\gamma)x,x) + \delta \begin{bmatrix} u_2(y^{\xi}(z;x),z) - \Phi_2\left(y^{\xi}(z;x),z\right) \\ -\Phi_1\left(y^{\xi}(z;x),z\right) \max\{0,(1-\gamma)\} \\ -\delta\Phi_1\left(y^{\xi}(y^{\xi}(z;x);z), y^{\xi}(z;x)\right) y_2^{\xi}(y^{\xi}(z;x);z) \end{bmatrix} \\ &\leq u_1(z-(1-\gamma)x,x) + \delta \begin{bmatrix} u_2(y^{\xi}(z;x),z) - \Phi_2\left(y^{\xi}(z;x),z\right) \\ -\Phi_1\left(y^{\xi}(z;x),z\right) y_1^{\xi}(z;x) \\ -\delta\Phi_1\left(y^{\xi}(y^{\xi}(z;x);z), y^{\xi}(z;x)\right) y_2^{\xi}(y^{\xi}(z;x);z) \end{bmatrix} = 0 \end{aligned}$$

We conclude that there is a $\xi^*$ such that for $\xi \leq \xi^*$, $P_{\xi}(z,x)$ is maximal in $[\underline{x}, \overline{x}]$ for all $x \in X$.

We verified numerically that $\xi^*$ for (36). Since $Y^{\xi}(z,x) \in [\underline{x}, \overline{x}]$ when $Y^{\xi}(z,x) \geq (1-\gamma)x - l$ and $Y^{\xi}(z,x) = (1-\gamma)x - l$ otherwise, this implies that $Y^{\xi}(z,x)$ is a optimal policy for all $x, z \in X$.

#### 8.2.2 The example in Panels B.1 and B.2

For the construction of the equilibrium in Panels B.1 and B.2 in Figure 3, we proceed in a similar way as in Section 3.2.1. For the limit case with $\xi = 0$, consider the equilibrium $y^*(x,\kappa)$ with $\kappa_R = c(\mu_R, a, b)$, where $c(\mu, a, b)$ is defined by (13) in the paper and $\mu_R$ is the only real solution to the equation $(\mu_R - 2)^2(\mu_R + 2) = 16$. The function $y(x, \kappa_R)$ is topologically conjugated with $\mu_R x(1-x)$ and, as proven by Ruelle [1977], has a chaotic trajectory in a set $X^*$. To construct an the equilibrium with $\xi > 0$ and similar properties, we look for a solution of (38) that is sufficiently close to $y(x, \kappa_R)$. To this goal, we exploit the fact that $y(x, \kappa_R)$ has an unstable cycle of period 2 with periodic points $x_1$, $x_2$ that can be found solving $[y]^2(x, \kappa_R) = x$. We therefore solve (38) approximating the solution as a polynomial of $x$ and $z$ and imposing the constraints that $p_{m_1,m_2}(x_1, x_2; \xi) = x_2$ and $p_{m_1,m_2}(x_2, x_1; \xi) = x_1$. A solution is found with high precision (in the order of $10^{-25}$) and it is used to construct the equilibrium strategy $Y^{\xi}(z,x)$ as in (37). We then verify that this function is an equilibrium following the same procedure as in Steps 1 and 2 in Section 3.2.1. The equilibrium described by $Y^{\xi}(z,x)$ computed in this way is not guaranteed to have chaotic dynamics in the formal sense described in Sections 2-3, but it generates the dynamics described in Panels B.1 and B.2 in Figure 3 in the paper.